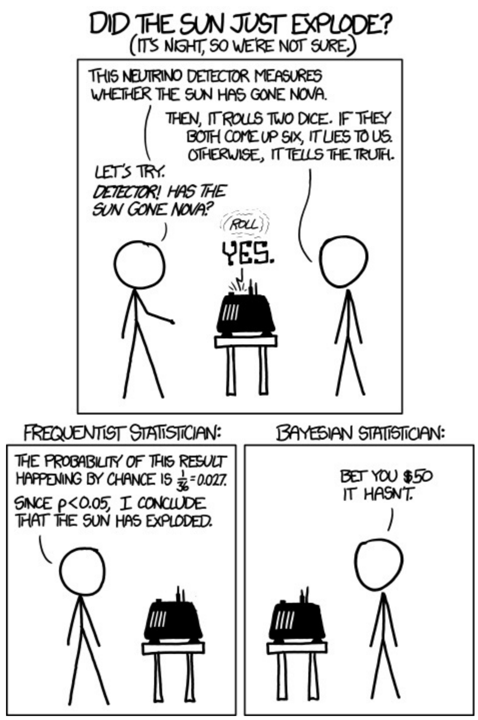
CSCI 104L Lecture 14: Bayes Theorem and Expectations

Question 1. There are two boxes. The first contains two gold balls and seven cardinal balls. The second contains four gold balls and three cardinal balls. You choose a box at random, then you choose a ball at random. You draw out a cardinal ball. What is the probability that the ball came from the first box?

> Given two events E and F with non-zero probability, then:
>
> $$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

Question 2. There is a rare disease which infects only 1 out of $100,000$ people. You can detect it with a very accurate diagnostic test. If someone has the disease, it correctly identifies it 99% of the time. If someone does not have the disease, it correctly states this 99.5% of the time.

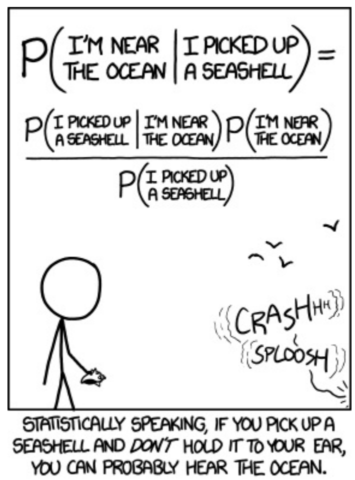
- Suppose the test comes out negative. What is the probability the person does not have the disease?
- Suppose the test comes out positive. What is the probability the person does have the disease?



XKCD # 1132: Frequentists vs. Bayesians. “Detector! What would the Bayesian statistician say if I asked him whether the–” [roll] “I am a neutrino detector, not a labyrinth guard. Seriously, did your brain fall out?” [roll] “. . . yes.”

Expected Value

The **Expected Value** of a random variable X is $E(X) = \sum_{s \in S} p(s)X(s)$. That is, the average value it will output.



XKCD # 1236: Seashell. This is roughly equivalent to 'number of times I've picked up a seashell at the ocean' / 'number of times I've picked up a seashell', which in my case is pretty close to 1, and gets much closer if we're considering only times I didn't put it to my ear.

Question 3. What is the expected value of a fair six-sided die?

Question 4. Given n Bernoulli trials each with probability of success p, what is the expected number of successes?

Linearity of Expectations: $E(X_1+X_2+...+X_n) = E(X_1)+E(X_2)+...+E(X_n)$. Also, $E(aX+b) = aE(X)+b$.

Question 5. What is the expected value of the sum of three fair six-sided dice?

Question 6. Professor Slacker hates grading, and just assigned everyone a uniform at random final course grade of A, B, C, or D. If he had actually bothered to grade, he still would have assigned an equal distribution of A's, B's, C's, and D's. What percentage of students actually receive the correct grade in expectation?

Question 7. The ordered pair (i, j) is called an *inversion* in a permutation of the first n positive integers if $i < j$ but j preceeds i in the permutation. For example, there are six inversions in the permutation $3, 5, 1, 4, 2$.

What is the expected number of inversions in a permutation of the first n positive integers, assuming all permutations are equally likely?

Question 8. What is the average number of comparisons used by InsertionSort to sort an array of n distinct elements?

To solve this, let's begin with allowing X to be the random variable equal to the number of comparisons used by the algorithm. We can separate it into a sum of X_i values, where X_i is the random variable equal to the number of comparisons used to insert a_i into the proper position after the first $i-1$ elements have been sorted.

Accordingly, $X = X_2 + X_3 + \ldots + X_n$ Similarly, $E(X) = E(X_2) + E(X_3) + \ldots + E(X_n)$

We now need only to determine each $E(X_i)$.

$E(X_1) = 0$

$E(X_2) = 1$

$E(X_3) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 2$

$E(X_4) = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot 3$

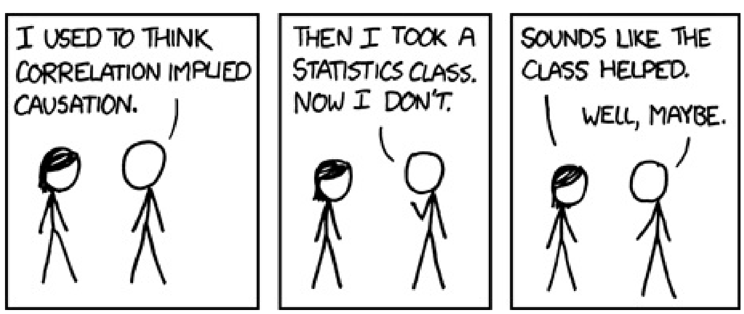
$E(X_i) = \frac{i-1}{i} + \frac{1}{i} \cdot \sum_{j=1}^{i-1} i$

Question 9. A coin comes up heads with probability p. We will flip this coin until it comes up heads. What is the expected number of flips required?

A random variable has a **geometric distribution** with parameter $p : 0 \leq p \leq 1$ if $p(X = k) = (1-p)^{k-1}p$, for $k = 1, 2, 3, \ldots$. It has expected value $\frac{1}{p}$.

Question 10. The probability that a randomly chosen 1000 digit number is prime is approximately 1/2302. Suppose we select a 1000 digit number at random and check if it is prime; if it is, we're done, and otherwise we randomly select another (possibly different) number and continue. What is the expected number of times we select a 1000 digit number?

Question 11. A medieval gladiator fights in an arena. On each day, he has a $\frac{1}{8}$ probability of dying, a $\frac{1}{4}$ probability of winning his freedom, and otherwise is sent back to his jail cell to fight again the next day. What is the expected number of days the gladiator will fight?



XKCD # 552: Correlation doesn't imply causation, but it does waggle its eyebrows suggestively and gesture furtively while mouthing 'look over there'.