CSCI 104L Lecture 11: Advanced Counting

Question 1. In how many ways can we select 3 students from a group of 5 students to stand in line for a picture?

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An **r-permutation** is an ordered arrangement of r elements of a set. If the set contains n elements, this is denoted as $P(n, r)$.

If n and r are integers with $0 \le r \le n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Question 2. How many permutations of the letters `DIJKSTRA` contain the substring `IJK`?

An **r-combination** is an unordered arrangement of r elements of a set. If the set contains n elements, this is denoted as $C(n, r)$. This can also be denoted by $\binom{n}{r}$, and called "n choose r".

$C(n, r) = \frac{n!}{r!(n-r)!}$

Question 3. There are 7 contestants, and you will select 3 winners (no distinction amongst them). How many different ways are there to do this?

A specific unordered arrangement of r elements can be ordered in $P(r, r) = r!$ ways. Thus $C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$

Question 4. How many different 5-card poker hands can be dealt from a standard deck of 52 cards?

Question 5. How many different 47-card hands can be dealt from a standard deck of 52 cards?

Question 6. How many bit strings of length n contain exactly r 1's?

$(x + y)^3 =?$

Binomial Theorem:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

This is called the **Binomial Theorem**. For this reason, we sometimes call $C(n, r)$ a binomial coefficient.

Question 7. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Proof: Suppose T is a set with $n + 1$ elements. Let a be an element in T, and let $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements, which consists of some number of elements with a and some number without a. There are $\binom{n}{k}$ ways to choose k elements from S, which corresponds to the subsets of T without a. There are $\binom{n}{k-1}$ ways to choose $k - 1$ elements from S, and we can add a to each of these, corresponding to the subsets of T with a.

Advanced Counting

Question 8. A message on a Twitter-like service consists of exactly r characters. Because it is the internet, people only communicate using capital letters, spaces, and the exclamation mark. How many different messages can one post on this Twitter-like service?

The number of r-permutations of a set of n objects, where repetition is allowed is n^r.

Question 9. How many different strings can be made by reordering the letters of the word SUCCESS?

The number of different permutations of n objects, where n_1 of them are indistinguishable objects of type 1, n_2 of them are indistinguishable objects of type 2, ..., and n_k of them are indistinguishable objects of type k is $\frac{n!}{n_1!n_2!...n_k!}$

Question 10. 4 players are playing Poker, wherein each player receives 5 cards. How many different ways are there to deal out hands?

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i is $\frac{n!}{n_1!n_2!...n_k!}$.

Question 11. How many ways are there to select five bills from a cash box containing $1, 2, 5, 10, 20, 50$ and 100 dollar bills?

The number of r-combinations of a set of n objects, where repetition is allowed is $C(n + r - 1, r)$.

Question 12. There are 8 poker players, and some of them have 100 dollar chips. There are a total of 10 one-hundred dollar chips distributed among the players. How many different ways are there to distribute the chips?

The number of ways to distribute r indistinguishable objects into n distinguishable boxes is $C(n + r - 1, r)$.
Question 13. How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears?

Question 14. How many solutions does the equation $x_1+x_2+x_3 = 11$ have, where x_1, x_2, x_3 are nonnegative integers?

It is very difficult to solve when the boxes are indistinguishable, unless the numbers are small.

Question 15. 4 different roommates are all playing a multi-player computer game, and they're distributed over 3 indistinguishable servers. How many ways are there to distribute them?

Question 16. Now consider the same problem where the players are faceless, such as from the game company's perspective. They only track statistics, they don't care who specifically plays where. How many ways are there to distribute 6 faceless players among 4 indistinguishable servers?

There is no simple formula for this, the only thing you can do is enumerate them.