1. A website requires that customers select a password that is a sequence of upper-case letters, lower-case letters and digits. What is the probability that a randomly selected string with exactly ten characters results in a password that has at least one upper-case letter, at least one lower-case letter, and at least one digit?
Solution:
Let P be the event that the password does not contain an upper-case character, L be the event that the password does not contain a lower-case character and D be the event that password does not contain a digit. The goal is to determine $p(\overline{P \cup L \cup D}) = 1 - p(P \cup L \cup D)$By inclusion exclusion:

$$p(P \cup L \cup D) = p(P) + p(L) + p(D) - p(P \cap L) - p(P \cap D) - p(L \cap D) + p(P \cap L \cap D)$$

The size of the sample space is $(26 + 26 + 10)^{10}$. If a password does not contain an upper-case letter, then it is composed entirely of lower-case letters and digits. There are $(10+26)^{10}$ such passwords. The same is true for passwords that do not include a lower-case letter. The number of passwords that do not contain a digit is $(26 + 26)^{10}$.

If a password does not contain upper-case letters and does not contain lower-case letters, then the password is composed entirely of digits. There are $(10)^{10}$ such passwords. A password that does not contain lower-case letters and does not contain digits is composed entirely of upper-case letters. There are $(26)^{10}$ such passwords. The same number holds for passwords that do not contain upper-case letters or digits.

Finally, there are no passwords that do not have any upper-case letters, lower-case letters or digits because those are the only characters allowed in the password and the password must have some characters.

$$p(P \cup L \cup D) = 2 \cdot \frac{(36)^{10}}{(62)^{10}} + \frac{(52)^{10}}{(62)^{10}} - 2 \cdot \frac{(26)^{10}}{(62)^{10}} - \frac{(10)^{10}}{(62)^{10}}$$

$$p(\overline{P \cup L \cup D}) = 1 - 2 \cdot (36/62)^{10} - (52/62)^{10} + 2 \cdot (26/62)^{10} + (10/62)^{10}$$

2. Backgammon is played with 15 indistinguishable black pieces and 15 indistinguishable white pieces on a board with 24 distinguishable spaces. Any number of pieces (including zero) may occupy each space, but black and white pieces cannot co-occupy the same space. Assume that all 30 pieces are on the board.

   (a) Assume that the 15 black pieces are distributed across exactly k spaces, and so the white pieces may only use the remaining $n - k$ spaces. How many backgammon positions exist?
   Solution:
   First, select the k spaces which are occupied by black pieces. There are $\binom{24}{k}$ ways to do this.
   Next, distribute the 15 black pieces across those k spaces. Each space must contain at least 1 black piece. Therefore, the problem simplifies to distributing $(15 - k)$ pieces across k spaces. Using the stars-and-bars method, there are $\binom{15-k+k-1}{k-1} = \binom{14}{k-1}$ ways to do this.
   Finally, distribute the 15 white pieces across the remaining $(24 - k)$ spaces. Using the stars-and-bars method, there are $\binom{15+24-k-1}{15} = \binom{38-k}{15}$ ways to do this.
   So, the total number of positions meeting the constraint is:

   $$\binom{24}{k}\binom{14}{15-k}\binom{38-k}{15}$$

   (b) Using your previous answer, find an expression for the total number of possible backgammon positions. Your expression may include a summation.
   Solution:
   $1 \leq k \leq 15$. So, the total number of positions is:

   $$\sum_{k=1}^{15}\binom{24}{k}\binom{14}{15-k}\binom{38-k}{15}$$

3. You throw a fair die. If the top face is even, you will earn dollars that are equal to the number on the top face. If the top face is odd, you need to pay dollars that are equal to the number on the top face. What is the expected dollars you will earn or pay?
Solution:
Let X be the random variable that represents the amount of dollars you will earn or pay.
When the top face is even, X will be 2, 4, 6.
When the top face is odd, X will be -1, -3, -5.
So the expectation is $E(X) = \frac{1}{6}(-1+2-3+4-5+6) = \frac{1}{2}$

4. A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3, and you are trying to determine whether his coin is fair. To accomplish this, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

   (a) What is the probability you reach an incorrect conclusion if the coin is fair (that is, a false negative)?
   Solution:
   The incorrect conclusion is reached if there are fewer than four heads. The probability that there are $0, 1, 2,$ or 3 heads is

$$\left(\frac{1}{2}\right)^{10} + 10 \cdot \left(\frac{1}{2}\right)^{10} + \binom{10}{2}\left(\frac{1}{2}\right)^{10} + \binom{10}{3}\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}(1 + 10 + 45 + 120) \approx 0.172$$

   (b) What is the probability that you reach an incorrect conclusion if the coin is biased (that is, a false positive)?
   Solution:
   The correct conclusion is reached if there are fewer than four heads. The probability that there are $0, 1, 2,$ or 3 heads is

$$(0.7)^{10} + \binom{10}{1}(0.7)^9(0.3) + \binom{10}{2}(0.7)^8(0.3)^2 + \binom{10}{3}(0.7)^7(0.3)^3 \approx 0.65$$

   The probability that the incorrect conclusion is reached is approximately 0.35.

5. The sensitivity of a diagnostic test is the probability that a true positive (e.g. someone with the disease under question) is correctly identified as such. The specificity, on the other hand, is the probability that a true negative (e.g. a healthy person) is not reported as a positive. You have a COVID-19 antibody test with specificity α and sensitivity β. There are N people in a particular population, and suppose we already know that n of them have the disease.

   (a) What is the probability that someone picked out of this population will be correctly diagnosed (either as a positive or a negative)?
   Solution:

$$\begin{aligned}
P[positive|disease] = \beta, \quad P[negative|healthy] = \alpha. \\
P[disease] = \frac{n}{N}, \quad P[healthy] = \frac{N-n}{N}. \\
P[correct] &= P[positive|disease]P[disease] + P[negative|healthy]P[healthy] \\
&= \beta\frac{n}{N} + \alpha\frac{N-n}{N}.
\end{aligned}$$

   (b) What is the probability that a positive diagnosis means someone actually has the disease? Or that a negative diagnosis means they don't have it?

Solution:
Bayes' rule!

$$\begin{aligned} P[disease|positive] &= \frac{P[positive|disease]P[disease]}{P[positive]} \\ &= \frac{P[positive|disease]P[disease]}{P[positive|disease]P[disease] + P[positive|healthy]P[healthy]} \\ &= \frac{\beta\frac{n}{N}}{\beta\frac{n}{N} + (1-\alpha)\frac{N-n}{N}} \end{aligned}$$

Likewise...

$$\begin{aligned} P[healthy|negative] &= \frac{P[negative|healthy]P[healthy]}{P[negative]} \\ &= \frac{P[negative|healthy]P[healthy]}{P[negative|healthy]P[healthy] + P[negative|disease]P[disease]} \\ &= \frac{\alpha\frac{N-n}{N}}{\alpha\frac{N-n}{N} + (1-\beta)\frac{n}{N}} \end{aligned}$$

(c) Only ten people out of a sampled population of 10,000 actually have the disease. Let $\alpha = 0.97$ and $\beta = 0.95$. How many false positives and false negatives are expected?
Solution:

$$\begin{aligned} P[disease|positive] &= \frac{0.95\frac{10}{10000}}{0.95\frac{10}{10000} + (1-0.97)\frac{9990}{10000}} \approx 3.1\% \\ P[healthy|positive] &= 1 - P[disease|positive] \approx 96.9\%. \end{aligned}$$

Although it turns out to be much easier to garner the false positives and false negatives! False positives:

$$9990 \times (1-\alpha) \approx 300$$

and false negatives:

$$10 \times (1-\beta) = 0.5$$

rounding up to maybe one expected false negative.

(d) Now let's say 9,000 actually have the disease. How many false positives and false negatives are expected this time?
Solution:
Similarly,

$$1000 \times (1-\alpha) = 30$$

false positives, and

$$9000 \times (1-\beta) = 450$$

false negatives in expectation.

6. How many integers n, $1 \leq n \leq 100$, are there such that $n^2 + 4n + 3$ is divisible by 7?
Solution:

$$n^2 + 4n + 3 \equiv 0 \pmod{7}$$

$$(n + 1)(n + 3) \equiv 0 \pmod{7}$$

In other words, $(n + 1)(n + 3)$ is a multiple of 7. Since 7 is prime, this requires that either $(n + 1)$ or $(n + 3)$ is a multiple of 7. So, either $(n + 1)$ or $(n + 3)$ is one of $\{7, 14, 21, ..., 98\}$. That makes $2 \cdot \lfloor \frac{100}{7} \rfloor = 28$ possible values of n.

Specifically, $n \in \{4, 6, 11, 13, 18, 20, ...\}$.

7. Suppose you have 50 gold balls, 50 cardinal balls, and 2 boxes. You can distribute the balls between the two boxes in any way, as long as neither box is empty. Then you will pick a box uniformly at random, and then pick a ball from that jar (again, uniformly at random).

   (a) How should you distribute the marbles to maximize the probability of choosing a gold ball?
   Solution:
   Place a single gold ball in box 1, and place all 99 other balls in box 2.

   (b) How should you distribute the marbles so that the probability of choosing a gold ball is independent of the box chosen?
   Solution:
   This will work as long as you have an equal number of gold and cardinal balls in each box. Then the probability is always 50%, regardless of box chosen.