1. A website requires that customers select a password that is a sequence of upper-case letters, lower-case letters and digits. What is the probability that a randomly selected string with exactly ten characters results in a password that has at least one upper-case letter, at least one lower-case letter, and at least one digit?

2. Backgammon is played with 15 indistinguishable black pieces and 15 indistinguishable white pieces on a board with 24 distinguishable spaces. Any number of pieces (including zero) may occupy each space, but black and white pieces cannot co-occupy the same space. Assume that all 30 pieces are on the board.

   (a) Assume that the 15 black pieces are distributed across exactly k spaces, and so the white pieces may only use the remaining $n - k$ spaces. How many backgammon positions exist?

   (b) Using your previous answer, find an expression for the total number of possible backgammon positions. Your expression may include a summation.

3. You throw a fair die. If the top face is even, you will earn dollars that are equal to the number on the top face. If the top face is odd, you need to pay dollars that are equal to the number on the top face. What is the expected dollars you will earn or pay?

4. A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3, and you are trying to determine whether his coin is fair. To accomplish this, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

   (a) What is the probability you reach an incorrect conclusion if the coin is fair (that is, a false negative)?

   (b) What is the probability that you reach an incorrect conclusion if the coin is biased (that is, a false positive)?

5. The sensitivity of a diagnostic test is the probability that a true positive (e.g. someone with the disease under question) is correctly identified as such. The specificity, on the other hand, is the probability that a true negative (e.g. a healthy person) is not reported as a positive. You have a COVID-19 antibody test with specificity α and sensitivity β. There are N people in a particular population, and suppose we already know that n of them have the disease.

   (a) What is the probability that someone picked out of this population will be correctly diagnosed (either as a positive or a negative)?

   (b) What is the probability that a positive diagnosis means someone actually has the disease? Or that a negative diagnosis means they don't have it?

   (c) Only ten people out of a sampled population of 10,000 actually have the disease. Let $\alpha = 0.97$ and $\beta = 0.95$. How many false positives and false negatives are expected?

   (d) Now let's say 9,000 actually have the disease. How many false positives and false negatives are expected this time?

6. How many integers n, $1 \leq n \leq 100$, are there such that $n^2 + 4n + 3$ is divisible by 7?

7. Suppose you have 50 gold balls, 50 cardinal balls, and 2 boxes. You can distribute the balls between the two boxes in any way, as long as neither box is empty. Then you will pick a box uniformly at random, and then pick a ball from that jar (again, uniformly at random).

   (a) How should you distribute the marbles to maximize the probability of choosing a gold ball?

   (b) How should you distribute the marbles so that the probability of choosing a gold ball is independent of the box chosen?