Unit 9

Implementing Combinational Functions with Karnaugh Maps or Memories

Outcomes

- I can use Karnaugh maps to synthesize combinational functions with several outputs
- I can determine the appropriate size and contents of a memory to implement any logic function (i.e. truth table)

A new way to synthesize your logic functions

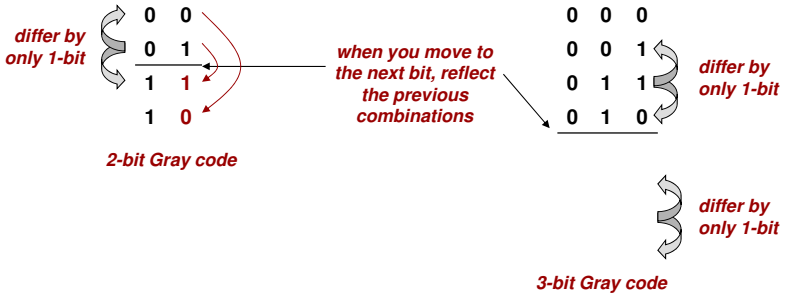
KARNAUGH MAPS

Logic Function Synthesis

- Given a function description as a T.T. or canonical form, how can we arrive at a circuit implementation or equation (i.e. perform logic synthesis)?
- First method
  - Minterms / maxterms
    - Can simplify to find minimal 2-level implementation
    - Use a decoder + 1 gate per output
- New, second method
  - Karnaugh Maps
    - Minimal 2-level implementation (though not necessarily minimal 3-, 4-, … level implementation)

Gray Code

- Different than normal binary ordering
- Reflective code
  - When you add the (n+1)th bit, reflect all the previous n-bit combinations
- Consecutive code words differ by only 1-bit



Karnaugh Maps

- If used correctly, will always yield a minimal, _____ implementation
  - There may be a more minimal 3-level, 4-level, 5-level… implementation but K-maps produce the minimal two-level (SOP or POS) implementation
- Represent the truth table graphically as a series of adjacent _____ that allows a human to see where variables will cancel

Karnaugh Map Construction

- Every square represents 1 input combination
- Must label axes in Gray code order
- Fill in squares with given function values

$F=\Sigma_{XYZ}(1,4,5,6)$

Z \ XY	00	01	11	10
0	0	0	1	1
1	1	0	0	1

3 Variable Karnaugh Map

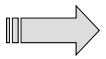
$G=\Sigma_{WXYZ}(1,2,3,5,6,7,9,10,11,14,15)$

YZ \ WX	00	01	11	10
00	0	0	0	0
01	1	1	0	1
11	1	1	1	1
10	1	1	1	1

4 Variable Karnaugh Map

Karnaugh Maps

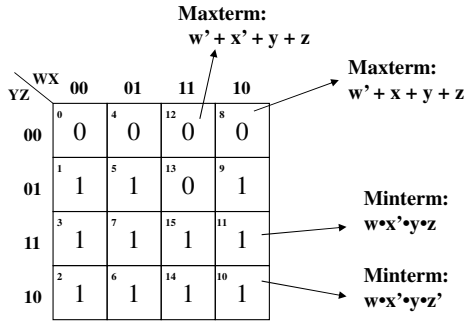
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



YZ \ WX	00	01	11	10
00	0	0	0	0
01	1	1	0	1
11	1	1	1	1
10	1	1	1	1

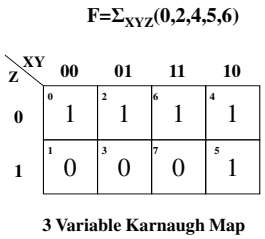
Karnaugh Maps

- Squares with a '1' represent minterms
- Squares with a '0' represent maxterms



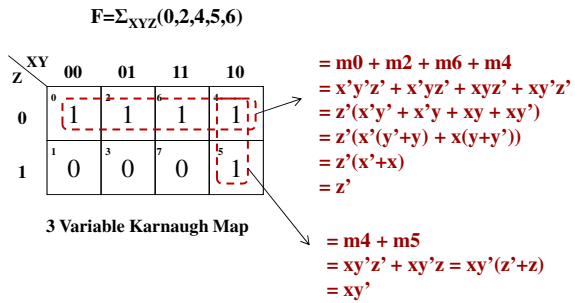
Karnaugh Maps

- Groups of adjacent 1's will always simplify to smaller product term than just individual minterms



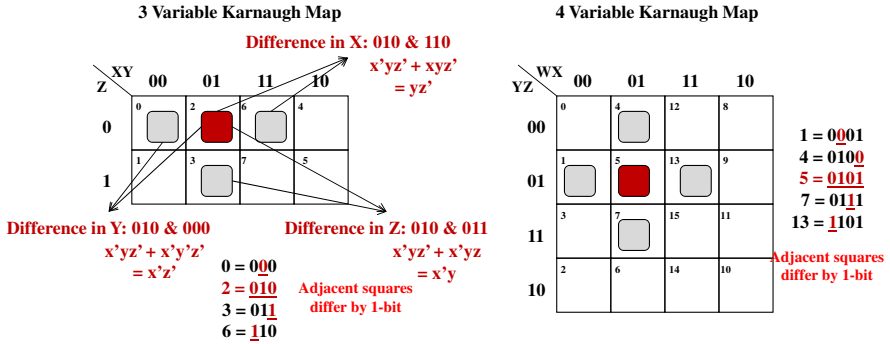
Karnaugh Maps

- Groups of adjacent 1's will always simplify to smaller product term than just individual minterms



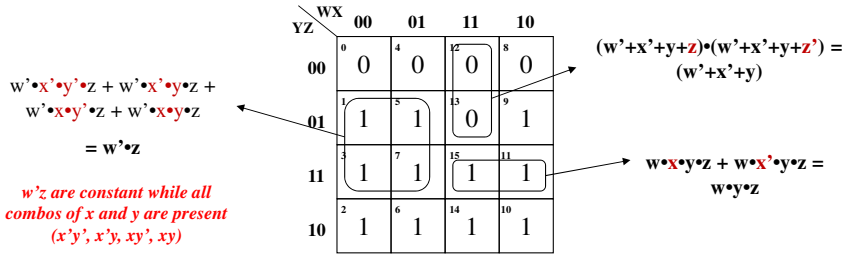
Karnaugh Maps

- Adjacent squares differ by 1-variable
  - This will allow us to use T10 = AB + AB'= A or T10' = (A+B')(A+B) = A



Karnaugh Maps

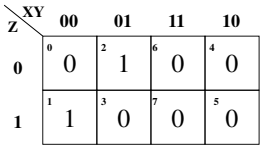
- 2 adjacent 1's (or 0's) differ by only one variable
- 4 adjacent 1's (or 0's) differ by two variables
- 8, 16, … adjacent 1's (or 0's) differ by 3, 4, … variables
- By grouping adjacent squares with 1's (or 0's) in them, we can come up with a simplified expression using T10 (or T10' for 0's)

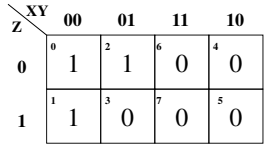


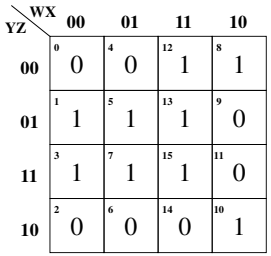
K-Map Grouping Rules

- Cover the 1's [=on-set] or 0's [=off-set] with _____ groups as possible, but make those groups _____ as possible
  - Make them as large as possible even if it means "covering" a 1 (or 0) that's already a member of another group
- Make groups of _____, … and they must be rectangular or square in shape.
- Wrapping is legal

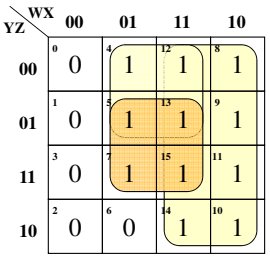
Group These K-Maps







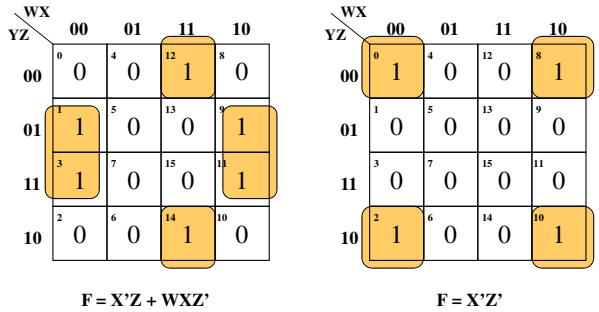
Karnaugh Maps



- Cover the remaining '1' with the largest group possible even if it "reuses" already covered 1's

Karnaugh Maps

- Groups can wrap around from:
  - Right to left
  - Top to bottom
  - Corners



Group This

YZ \ WX	00	01	11	10
00	0	0	0	0
01	1	1	0	1
11	1	1	1	1
10	1	1	1	1

K-Map Translation Rules

- When translating a group of 1's, find the variable values that are constant for each square in the group and translate only those variables values to a product term
- Grouping 1's yields SOP
- When translating a group of 0's, again find the variable values that are constant for each square in the group and translate only those variable values to a sum term
- Grouping 0's yields POS

Karnaugh Maps (SOP)

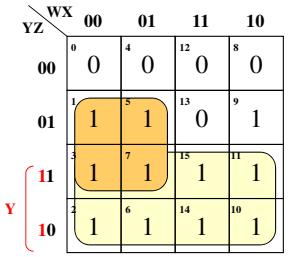
W	X	Y	Z	F
0	0	0	0	**0**
0	0	0	1	**1**
0	0	1	0	**1**
0	0	1	1	**1**
0	1	0	0	**0**
0	1	0	1	**1**
0	1	1	0	**1**
0	1	1	1	**1**
1	0	0	0	**0**
1	0	0	1	**1**
1	0	1	0	**1**
1	0	1	1	**1**
1	1	0	0	**0**
1	1	0	1	**0**
1	1	1	0	**1**
1	1	1	1	**1**

YZ \ WX	00	01	11	10
00	0	0	0	0
01	1	1	0	1
11	1	1	1	1
10	1	1	1	1

F =

Karnaugh Maps (SOP)

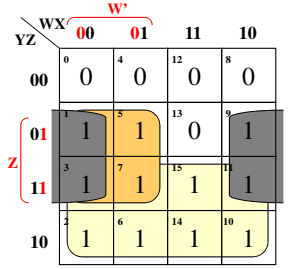
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



F = Y

Karnaugh Maps (SOP)

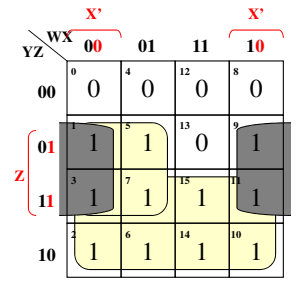
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



F = Y + W'Z + …

Karnaugh Maps (SOP)

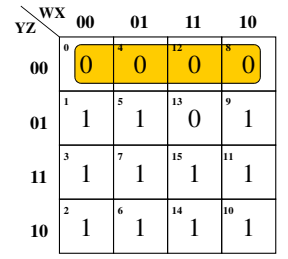
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



F = Y + W'Z + X'Z

Karnaugh Maps (POS)

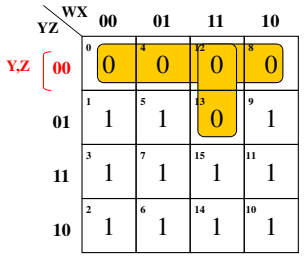
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



F =

Karnaugh Maps (POS)

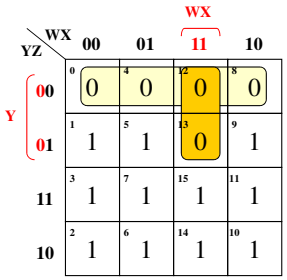
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



F = (Y+Z)

Karnaugh Maps (POS)

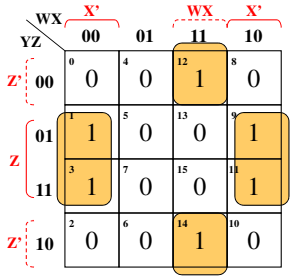
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



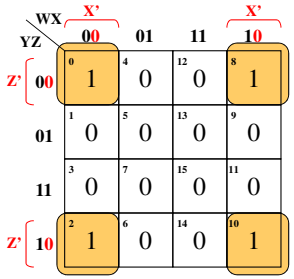
F = (Y+Z)(W'+X'+Y)

Karnaugh Maps

- Groups can wrap around from:
  - Right to left
  - Top to bottom
  - Corners

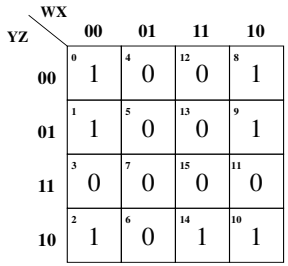


F = X'Z + WXZ'

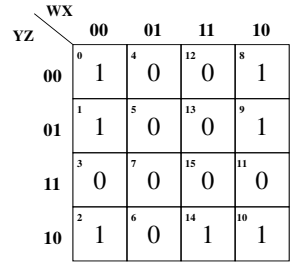


F = X'Z'

Exercises

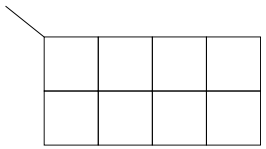


F_{SOP}=



F_{POS}=

$P=\Sigma_{XYZ}(2,3,5,7)$



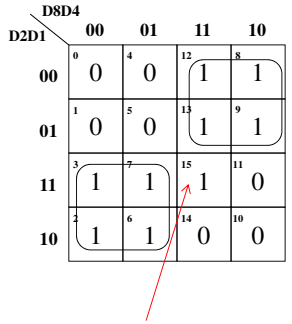
P=

No Redundant Groups

YZ \ WX	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	0	0	0	0
10	1	0	1	1

Multiple Minimal Expressions

- For some functions, _____ groupings exist which will lead to alternate minimal _____…Pick one



If time permits…

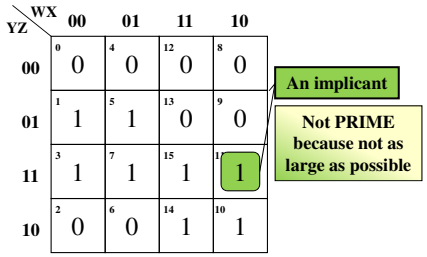
FORMAL TERMINOLOGY FOR KMAPS

Terminology

- Implicant: A product term (grouping of 1's) that covers a subset of cases where F=1
  - the product term is said to "imply" F because if the product term evaluates to '1' then F='1'
- Prime Implicant: The largest grouping of 1's (smallest product term) that can be made
- Essential Prime Implicant: A prime implicant (product term) that is needed to cover all the 1's of F

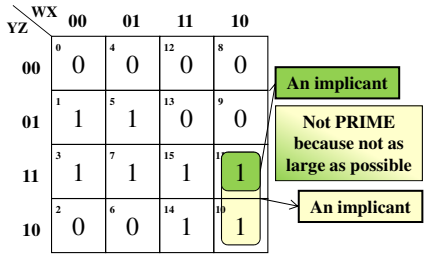
Implicant Examples

W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



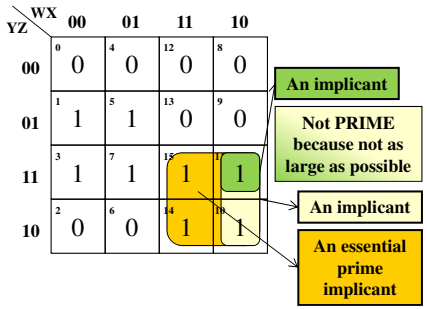
Implicant Examples

W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



Implicant Examples

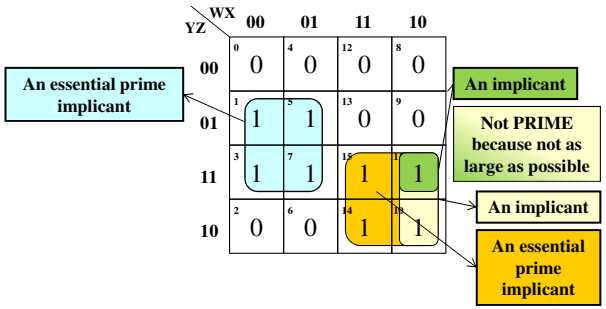
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



An essential prime implicant (largest grouping possible, that must be included to cover all 1's)

Implicant Examples

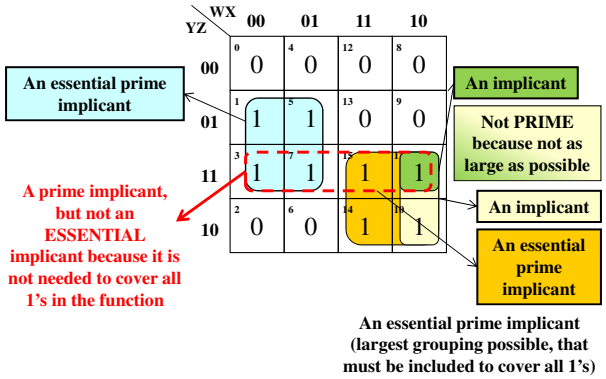
W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



An essential prime implicant (largest grouping possible, that must be included to cover all 1's)

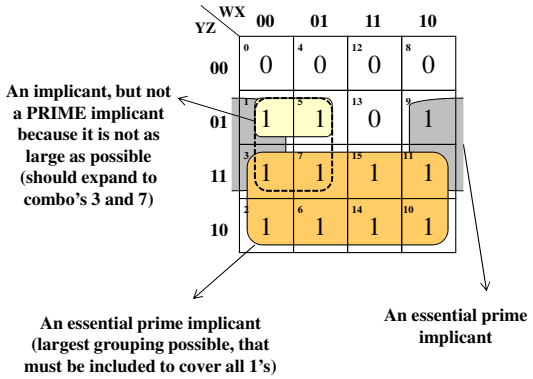
Implicant Examples

W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



Implicant Examples

W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1



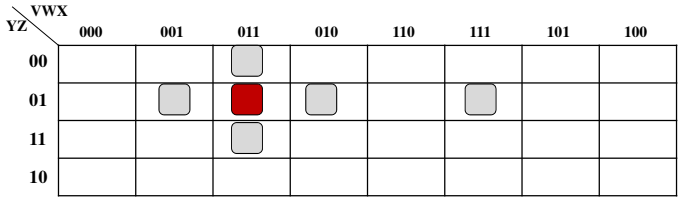
K-Map Grouping Rules

- Make groups (implicants) of 1, 2, 4, 8, ... and they must be rectangular or square in shape.
- Include the minimum number of essential prime implicants
  - Use only *essential* prime implicants (i.e. as few groups as possible to cover all 1's)
  - Ensure that you are using ***prime*** implicants (i.e. Always make groups as large as possible reusing squares if necessary)

5- & 6-VARIABLE KMAPS

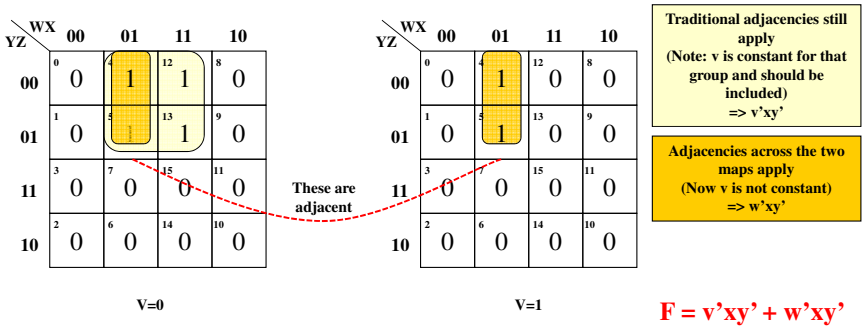
5-Variable K-Map

- If we have a 5-variable function we need a 32-square KMap.
- Will an 8x4 matrix work?
  - Recall K-maps work because adjacent squares differ by 1-bit
- How many adjacencies should we have for a given square?
- ___!! But drawn in 2 dimensions we can't have __ adjacencies.



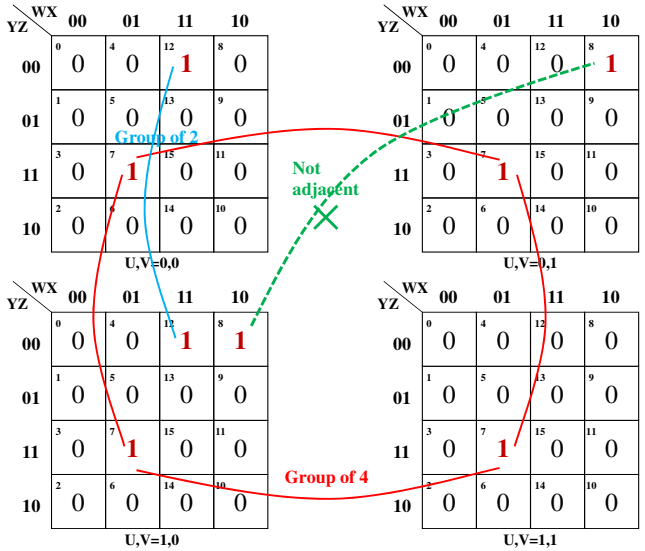
5-Variable Karnaugh Maps

- To represent the 5 adjacencies of a 5-variable function [e.g. f(v,w,x,y,z)], imagine two 4x4 K-Maps stacked on top of each other
  - Adjacency across the two maps



6-Variable Karnaugh Maps

- 6 adjacencies for 6-variables (Stack of four 4x4 maps)



DON'T CARE OUTPUTS

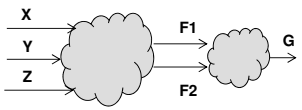
Don't-Cares

- Sometimes there are certain input combinations that are illegal (i.e. in BCD, 1010 – 1111 can never occur)
- The outputs for the illegal inputs are "don't-cares"
  - The output can either be 0 or 1 since the inputs can never occur
  - Don't-cares can be included in groups of 1 or groups of 0 when grouping in K-Maps
  - Use them to make as big of groups as possible

Use 'Don't care' outputs as wildcards (e.g. the blank tile in Scrabble™). They can be either 0 or 1 whatever helps make bigger groups to cover the ACTUAL 1's

Combining Functions

- Given intermediate functions F1 and F2, how could you use AND, OR, NOT to make G
- Notice certain F1,F2 combinations never occur in G(x,y,z)…what should we make their output in the T.T.



X	Y	Z	F1	F2	G
0	0	0	0	0	0
0	0	1	1	0	1
0	1	0	1	0	1
0	1	1	1	0	1
1	0	0	1	0	1
1	0	1	1	0	1
1	1	0	1	0	1
1	1	1	1	1	0

F1	F2	G
0	0	
0	1	
1	0	
1	1	

Don't Care Example

D8	D4	D2	D1	GT6
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	d
1	0	1	1	d
1	1	0	0	d
1	1	0	1	d
1	1	1	0	d
1	1	1	1	d

D8D4 / D2D1

D2D1 \ D8D4	00	01	11	10
00	0	0	d	1
01	0	0	d	1
11	0	1	d	d
10	0	0	d	d

$GT6_{SOP}$=

D8D4 / D2D1

D2D1 \ D8D4	00	01	11	10
00	0	0	d	1
01	0	0	d	1
11	0	1	d	d
10	0	0	d	d

$GT6_{POS}$=

Don't Cares

W	X	Y	Z	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	d
1	0	1	1	d
1	1	0	0	d
1	1	0	1	d
1	1	1	0	d
1	1	1	1	d

WX / YZ

YZ \ WX	00	01	11	10
00	0	0	d	0
01	1	1	d	1
11	1	1	d	d
10	1	1	d	d

F = Z + Y

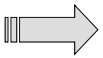
Reuse "d's" to make as large a group as possible to cover 1,5, & 9

Use these 4 "d's" to make a group of 8

Don't Cares

W	X	Y	Z	F
0	0	0	0	**0**
0	0	0	1	**1**
0	0	1	0	**1**
0	0	1	1	**1**
0	1	0	0	**0**
0	1	0	1	**1**
0	1	1	0	**1**
0	1	1	1	**1**
1	0	0	0	**0**
1	0	0	1	**1**
1	0	1	0	**d**
1	0	1	1	**d**
1	1	0	0	**d**
1	1	0	1	**d**
1	1	1	0	**d**
1	1	1	1	**d**

You can use "d's" when grouping 0's and converting to POS

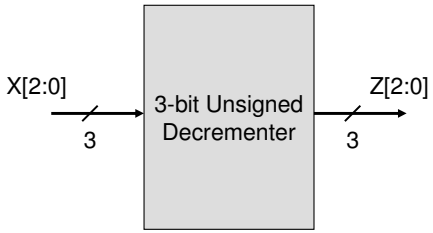


YZ \ WX	00	01	11	10
00	0	0	d	0
01	1	1	d	1
11	1	1	d	d
10	1	1	d	d

F = Y+Z

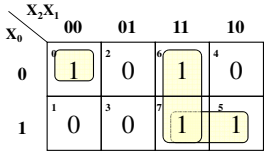
Designing Circuits w/ K-Maps

- Given a description…
  - Block Diagram
  - Truth Table
  - K-Map for each output bit (each output bit is a separate function of the inputs)
- 3-bit unsigned decrementer (Z = X-1)
  - If X[2:0] = 000 then Z[2:0] = 111, etc.



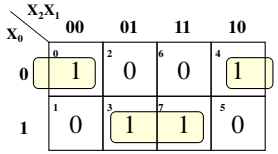
3-bit Number Decrementer

X_2	X_1	X_0	Z_2	Z_1	Z_0
0	0	0	**1**	**1**	**1**
0	0	1	**0**	**0**	**0**
0	1	0	**0**	**0**	**1**
0	1	1	**0**	**1**	**0**
1	0	0	**0**	**1**	**1**
1	0	1	**1**	**0**	**0**
1	1	0	**1**	**0**	**1**
1	1	1	**1**	**1**	**0**



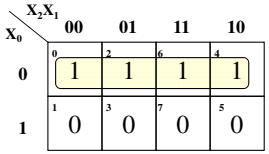
X_0 \ X_2X_1	00	01	11	10
0	1	0	1	0
1	0	0	1	1

$Z_2 = X_2X_0 + X_2X_1 + X_2'X_1'X_0'$



X_0 \ X_2X_1	00	01	11	10
0	1	0	0	1
1	0	1	1	0

$Z_1 = X_1'X_0' + X_1X_0$



X_0 \ X_2X_1	00	01	11	10
0	1	1	1	1
1	0	0	0	0

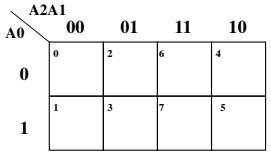
$Z_0 = X_0'$

Squaring Circuit

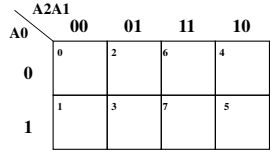
- Design a combinational circuit that accepts a 3-bit number and generates an output binary number equal to the square of the input number. ($B = A^2$)
- Using 3 bits we can represent the numbers from _____ to _____ .
- The possible squared values range from _____ to _____ .
- Thus to represent the possible outputs we need how many bits? _____

3-bit Squaring Circuit

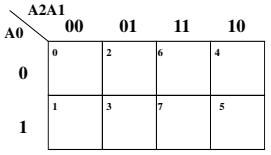
A	Inputs A_2	A_1	A_0	Outputs B_5	B_4	B_3	B_2	B_1	B_0	$B=A^2$



B5 =

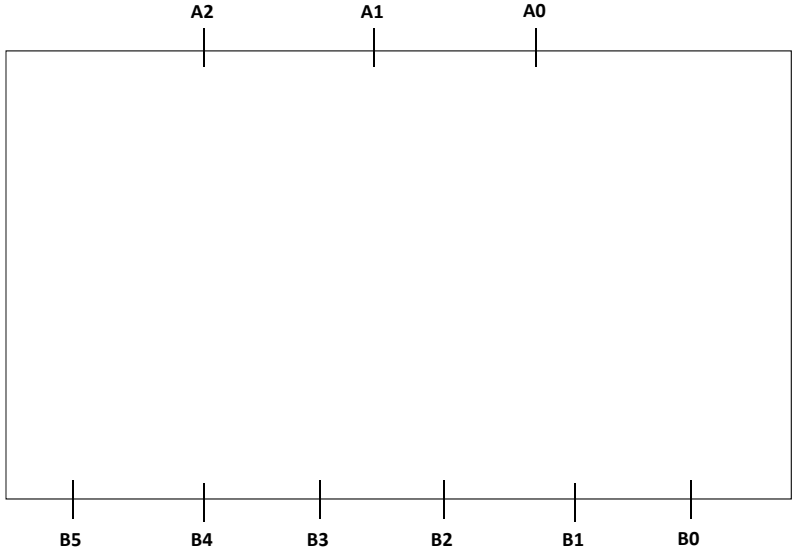


B4 =



B0 =

3-bit Squaring Circuit



USING MEMORIES TO BUILD COMBINATIONAL CIRCUITS

Dimensions and Operations

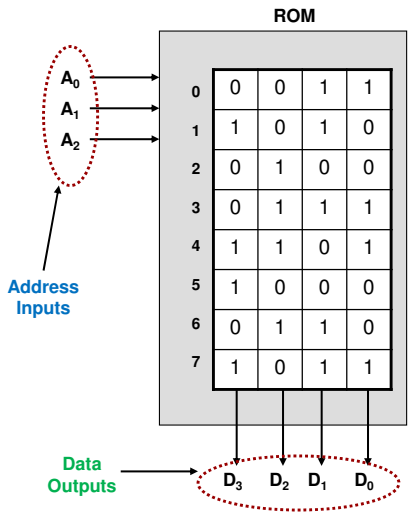
MEMORY BASICS

Memories

- Memories store (write) and retrieve (read) data
  - Read-Only Memories (ROM's): Can only retrieve data (contents are initialized and then cannot be changed)
  - Read-Write Memories (RWM's): Can retrieve data and change the contents to store new data

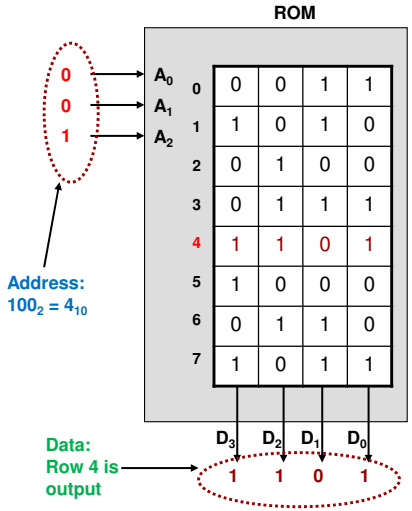
ROM's

- Memories are just _____ of data with rows and columns
- When data is read, one entire ____ of data is read out
- The row to be read is selected by putting a binary number on the _____ inputs



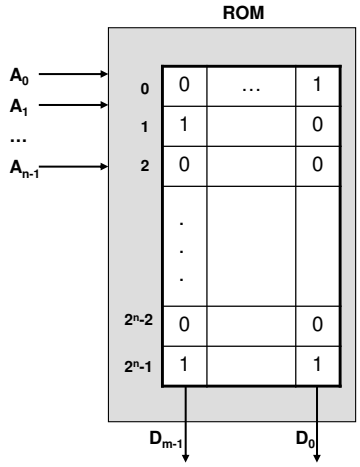
ROM's

- Example
  - Address = 4 dec. = 100 bin. is provided as input
  - ROM outputs data in that row (1101 bin.)



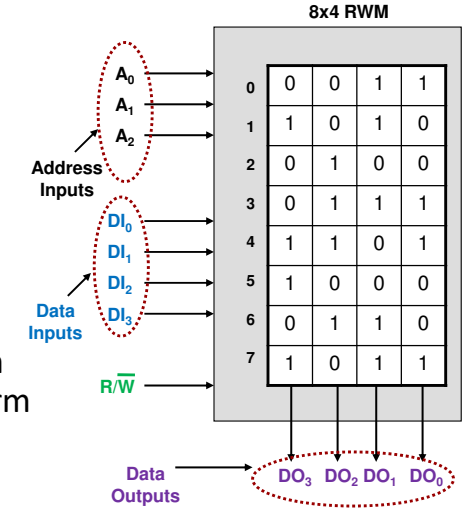
Memory Dimensions

- Memories are named by their dimensions:
  - Rows x Columns
- *n* rows and *m* columns => __ x ___ ROM
- 2^n rows => n address bits
  …or…
  k rows => $\log_2 k$ address bits
- m cols => m data outputs



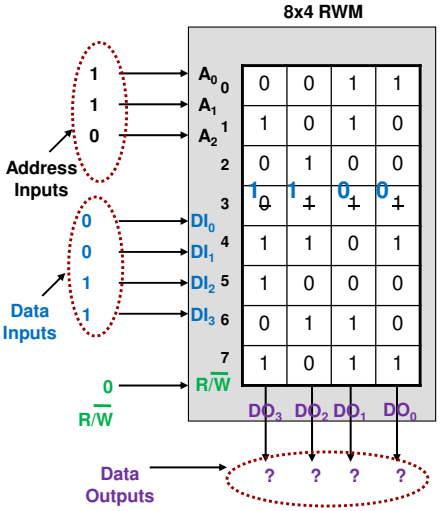
RWM's

- Writable memories provide a set of data inputs for write data (as opposed to the data outputs for read data)
- A control signal R/$\overline{W}$ (1=READ / 0 = WRITE) is provided to tell the memory what operation the user wants to perform



RWM's

- Write example
  - Address = 3 dec. = 011 bin.
  - DI = 12 dec. = 1100 bin.
  - R/$\overline{W}$ = 0 => Write op.
- Data in row 3 is overwritten with the new value of 1100 bin.



Look-up tables…

USING MEMORIES TO BUILD COMBINATIONAL FUNCTIONS

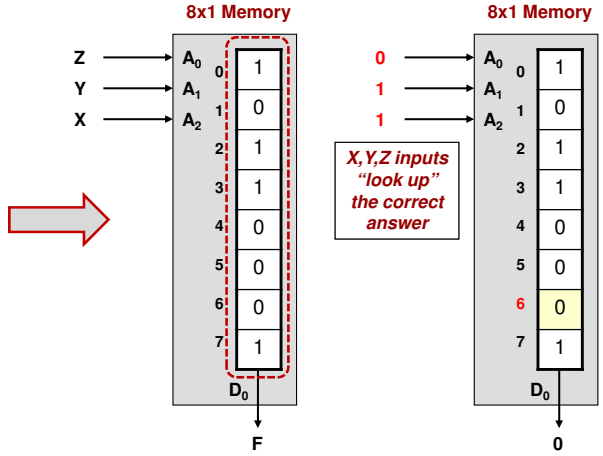
Memories as Look-Up Tables

- One major application of memories in digital design is to use them as LUT's (Look-Up Tables) to implement logic functions
  - This is the core technology used by FPGAs (Field-Programmable Gate Arrays) which are used pervasively in robotics, space applications, networking, and even in search engines
- **Idea**: Use a memory to hold the _____ of a function and feed the inputs of the function to the _____ inputs to "_____" the answer

Implementing Functions w/ Memories

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Arbitrary Logic Function

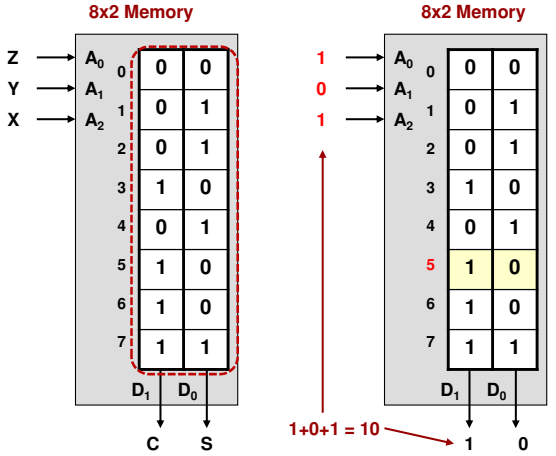


Use a memory with the same dimensions as 'output' side of the truth table. It's almost TOO easy.

Implementing Functions w/ Memories

X	Y	Z	C	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Multi-bit function (One's count)



Use a memory with the same dimensions as 'output' side of the truth table. It's almost TOO easy.

3-bit Squaring Circuit

- **Q:** What size memory would you use to build our 3-bit squaring circuit?
- **A:** ____ memory
- **Q:** What would you connect to the address inputs of the memory?
- **A:** _____
- **Q:** What bits would you program into row 5 of the memory?
- **A:** _____

A	Inputs A_2	A_1	A_0	Outputs B_5	B_4	B_3	B_2	B_1	B_0	$B=A^2$
0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	1	1
2	0	1	0	0	0	0	1	0	0	4
3	0	1	1	0	0	1	0	0	1	9
4	1	0	0	0	1	0	0	0	0	16
5	1	0	1	0	1	1	0	0	1	25
6	1	1	0	1	0	0	1	0	0	36
7	1	1	1	1	1	0	0	0	1	49

4x4 Multiplier Example

Determine the dimensions of the memory that would be necessary to implement a 4x4-bit unsigned multiplier with inputs X[3:0] and Y[3:0] and outputs P[??:0]

Question: How many bits are needed for P?

Question: What are the contents of the numbered rows?

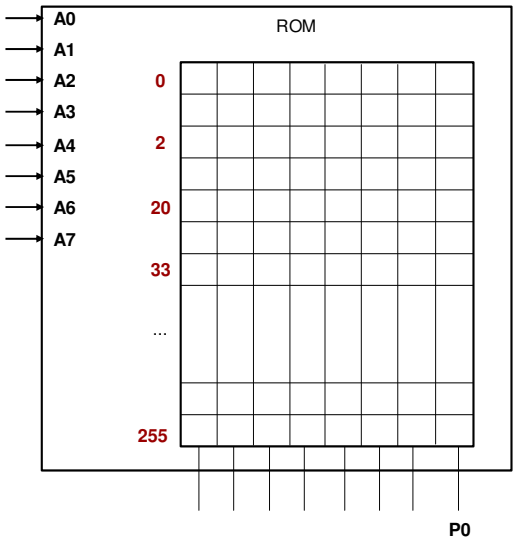
Example:

$X_3X_2X_1X_0$=0010

$Y_3Y_2Y_1Y_0$=0001

P = X * Y = 2 * 1 = 2

= 00010



Implementing Functions w/ Memories

- To implement a function w/ n-variables and m outputs
- Just place the output truth table values in the memory
- Memory will have dimensions: 2^n rows and m columns
  - Still does not scale terribly well (i.e. n-inputs requires memory w/ 2^n outputs)
  - But it is easy and since we can change the contents of memories it allows us to create "reconfigurable" logic
  - This idea is at the heart of FPGAs