Unit 3

Number Systems

Boolean Algebra Part 1

ANALOG VS. DIGITAL

Analog vs. Digital

- The analog world is based on continuous events. Observations can take on (real) any value.
- The digital world is based on discrete events. Observations can only take on a finite number of discrete values

Analog vs. Digital

- Q. Which is better?
- A. Depends on what you are trying to do.
- Some tasks are better handled with analog data, others with digital data.
  - Analog means continuous/real valued signals with an infinite number of possible values
  - Digital signals are discrete [i.e. 1 of n values]

Analog vs. Digital

- How much money is in my checking account?
  - Analog: Oh, some, but not too much.
  - Digital: $243.67

Analog vs. Digital

- How much do you love me?
  - Analog: I love you with all my heart!!!!
  - Digital: 3.2 x 10^3 MegaHearts

The Real (Analog) World

- The real world is inherently analog.
- To interface with it, our digital systems need to:
  - Convert analog signals to digital values (numbers) at the input.
  - Convert digital values to analog signals at the output.
- Analog signals can come in many forms
  - Voltage, current, light, color, magnetic fields, pressure, temperature, acceleration, orientation

Digital is About Numbers

- In a digital world, numbers are used to represent all the possible discrete events
  - Numerical values
  - Computer instructions (ADD, SUB, BLE, …)
  - Characters ('a', 'b', 'c', …)
  - Conditions (on, off, ready, paper jam, …)
- Numbers allow for easy manipulation
  - Add, multiply, compare, store, …
- Results are repeatable
  - Each time we add the same two number we get the same result

DIGITAL REPRESENTATION

Interpreting Binary Strings

- Given a string of 1's and 0's, you need to know the *representation system* being used, before you can understand the value of those 1's and 0's.
- ___________________________________

01000001 = ?

Binary Representation Systems

- Integer Systems
  - Unsigned
    - Unsigned (Normal) binary
  - Signed
    - Signed Magnitude
    - 2's complement
    - *Excess-N**
    - *1's complement**
- Floating Point
  - For very large and small (fractional) numbers
- Codes
  - Text
    - ASCII / Unicode
  - Decimal Codes
    - BCD (Binary Coded Decimal) / (8421 Code)

* = Not fully covered in this class

Number Systems

- Number systems consist of
  1. _______________
  2. ___ coefficients [__________]
- Human System: Decimal (Base 10): 0,1,2,3,4,5,6,7,8,9
- Computer System: Binary (Base 2): 0,1
- Human systems for working with computer systems (shorthand for human to read/write binary)
  - ______________________________
  - ______________________________

Anatomy of a Decimal Number

- A number consists of a string of explicit coefficients (digits).
- Each coefficient has an implicit place value which is a _______ of the base.
- The value of a decimal number (a string of decimal coefficients) is the sum of each coefficient times it place value

radix (base)

$(934)_{10}$ = 9*___ + 3*___ + 4*___ = _____

Explicit coefficients

Implicit place values

$(3.52)_{10}$ = 3*____ + 5*____ + 2*____ = ____

Anatomy of a Binary Number

- Same as decimal but now the coefficients are 1 and 0 and the place values are the powers of 2

Most Significant Digit (MSB)

Least Significant Bit (LSB)

$(1011)_2$ = 1*__ + 0*_ + 1*__ + 1*__

radix (base)

coefficients

place values = powers of 2

General Conversion From Base r to Decimal

- A number in base r has place values/weights that are the powers of the base
- Denote the coefficients as: a_i

$$(a_3a_2a_1a_0.a_{-1}a_{-2})_r = a_3*r^3 + a_2*r^2 + a_1*r^1 + a_0*r^0 + a_{-1}*r^{-1} + a_{-2}*r^{-2}$$

Left-most digit = Most Significant Digit (MSD)

Right-most digit = Least Significant Digit (LSD)

N_r => _________ => D_{10}

Number in base r

Decimal Equivalent

Examples

$(746)_8$ =

$(1A5)_{16}$ =

$(AD2)_{16}$ =

Binary Examples

$(1001.1)_2$ =

$(10110001)_2$ =

Powers of 2

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$

___	___	___	___	___	___	___	___	___	___	___
1024	512	256	128	64	32	16	8	4	2	1

Unique Combinations

- Given *n* digits of base *r*, how many unique numbers can be formed? __
  - What is the range? [________]

2-digit, decimal numbers (r=10, n=2) ___ ___ (0-9, 0-9)

3-digit, decimal numbers (r=10, n=3) ___ ___ ___

4-bit, binary numbers (r=2, n=4) ___ ___ ___ ___ (0-1, 0-1, 0-1, 0-1)

6-bit, binary numbers (r=2, n=6) ___ ___ ___ ___ ___ ___

Main Point: Given n digits of base r, ___ unique numbers can be made with the range [________]

Approximating Large Powers of 2

- Often need to find decimal approximation of a large powers of 2 like 2^{16}, 2^{32}, etc.
- Use following approximations:
  - $2^{10} \approx$ ________________
  - $2^{20} \approx$ ________________
  - $2^{30} \approx$ ________________
  - $2^{40} \approx$ ________________
- For other powers of 2, decompose into product of 2^{10} or 2^{20} or 2^{30} and a power of 2 that is less than 2^{10}
  - 16-bit half word: 64K numbers
  - 32-bit word: 4G numbers
  - 64-bit dword: 16 million trillion numbers

$2^{16} = 2^6 * 2^{10}$
$\approx$

2^{24} =

2^{28} =

2^{32} =

Decimal to Unsigned Binary

- To convert a decimal number, *x,* to binary:
  - Only coefficients of 1 or 0. So simply find place values that add up to the desired values, starting with larger place values and proceeding to smaller values and place a 1 in those place values and 0 in all others

25_{10} = ___ ___ ___ ___ ___ ___

32	16	8	4	2	1

Decimal to Unsigned Binary

73_{10}= ___ ___ ___ ___ ___ ___ ___ ___

128	64	32	16	8	4	2	1

87_{10}= ___ ___ ___ ___ ___ ___ ___ ___

145_{10}= ___ ___ ___ ___ ___ ___ ___ ___

0.625_{10}= ___ ___ ___ ___ ___

.5	.25	.125	.0625	.03125

Decimal to Another Base

- To convert a decimal number, *x,* to base r:
  - Use the place values of base r (powers of r). Starting with largest place values, fill in coefficients that sum up to desired decimal value without going over.

75_{10} = ___ ___ ___ hex

256	16	1

Hexadecimal and Octal

SHORTHAND FOR BINARY

Binary, Octal, and Hexadecimal

- Octal (base 8 = 2^3)
- 1 Octal digit (_)$_8$ can represent: ________
- 3 bits of binary (_ _ _)$_2$ can represent: 000-111 = ________
- Conclusion… __Octal digit = __ bits

- Hex (base 16=2^4)
- 1 Hex digit (_)$_{16}$ can represent: 0-F (_____)
- 4 bits of binary (_ _ _ _)$_2$ can represent: 0000-1111= ______
- Conclusion… __ Hex digit = ___ bits

Binary to Octal or Hex

- Make groups of 3 bits starting from radix point and working outward
- Add 0's where necessary
- Convert each group of 3 to an octal digit

101001110.11

- Make groups of 4 bits starting from radix point and working outward
- Add 0's where necessary
- Convert each group of 4 to an octal digit

101001110.11

Octal or Hex to Binary

- Expand each octal digit to a group of 3 bits

317.2_8

- Expand each hex digit to a group of 4 bits

$D93.8_{16}$

Hexadecimal Representation

- Since values in modern computers are many bits, we use hexadecimal as a shorthand notation (4 bits = 1 hex digit)
  - 11010010 = D2 hex or 0xD2 if you write it in C/C++
  - 0111011011001011 = 76CB hex or 0x76CB if you write it in C/C++

ASCII & Unicode

BINARY CODES

Binary Representation Systems

- Integer Systems
  - Unsigned
    - Unsigned (Normal) binary
  - Signed
    - Signed Magnitude
    - 2's complement
    - *1's complement**
    - *Excess-N**
- Floating Point
  - For very large and small (fractional) numbers
- Codes
  - Text
    - ASCII / Unicode
  - Decimal Codes
    - BCD (Binary Coded Decimal) / (8421 Code)

* = Not covered in this class

Binary Codes

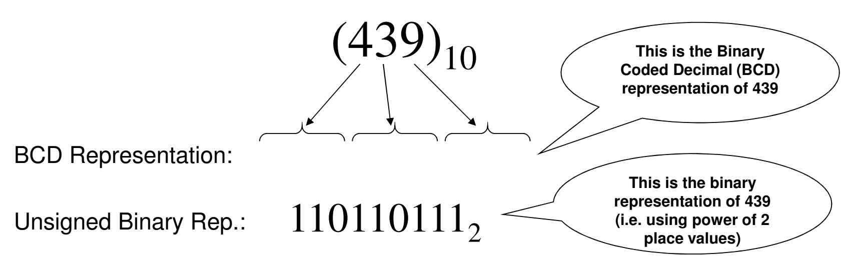
- Using binary we can represent any kind of information by coming up with a code
- Using *n* bits we can represent 2^n distinct items

Colors of the rainbow:
- Red = 000
- Orange = 001
- Yellow = 010
- Green = 100
- Blue = 101
- Purple = 111

Letters:
- 'A' = 00000
- 'B' = 00001
- 'C' = 00010
- .
- .
- .
- 'Z' = 11001

BCD

- Rather than convert a decimal number to binary which may lose some precision (i.e. 0.1_{10} = infinite binary fraction), BCD represents each decimal digit as a separate group of bits (exact decimal precision)
  - Each digits is represented as a ___________ number (using place values 8,4,2,1 for each dec. digit)
  - Often used in financial and other applications where decimal precision is needed



Important: Some processors have specific instructions to operate on #'s represented in BCD

ASCII Code

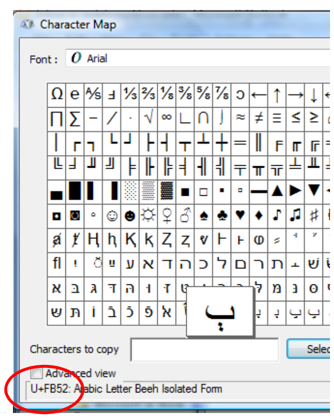
- Used for representing text characters
- Originally 7-bits but usually stored as 8-bits = 1-byte in a computer
- Example:
  - `"Hello\n";`
  - Each character is converted to ASCII equivalent
    - ‘H’ = 0x48, ‘e’ = 0x65, …
    - \n = newline character is represented by either one or two ASCII character

ASCII Table

LSD/MSD	0	1	2	3	4	5	6	7
0	NULL	DLW	SPACE	0	@	P	`	p
1	SOH	DC1	!	1	A	Q	a	q
2	STX	DC2	“	2	B	R	b	r
3	ETX	DC3	#	3	C	S	c	s
4	EOT	DC4	$	4	D	T	d	t
5	ENQ	NAK	%	5	E	U	e	u
6	ACK	SYN	&	6	F	V	f	v
7	BEL	ETB	‘	7	G	W	g	w
8	BS	CAN	(	8	H	X	h	x
9	TAB	EM	)	9	I	Y	i	y
A	LF	SUB	*	:	J	Z	j	z
B	VT	ESC	+	;	K	[	k	{
C	FF	FS	,	<	L	\	l	|
D	CR	GS	-	=	M	]	m	}
E	SO	RS	.	>	N	^	n	~
F	SI	US	/	?	O	_	o	DEL

UniCode

- ASCII can represent only the English alphabet, decimal digits, and punctuation
  - 7-bit code => 2^7 = _____ characters
  - It would be nice to have one code that represented more alphabets/characters for common languages used around the world
- Unicode
  - 16-bit Code => _______ characters
  - Represents many languages alphabets and characters
  - Used by Java as standard character code



Unicode hex value
(i.e. FB52 => 1111101101010010)

BOOLEAN ALGEBRA INTRO

Boolean Algebra

- A set of theorems to help us manipulate logical expressions/equations
- Axioms = Basis / assumptions used
- Theorems = manipulations that we can use

Axioms

- Axioms are the basis for Boolean Algebra
- Notice that these axioms are simply restating our definition of digital/binary logic
  - A1/A1' = ____________________
  - A2/A2' = ___________
  - A3,A4,A5 = _____________
  - A3',A4',A5' = _____________

(A1)	X = 0 if X ≠ 1	(A1')	X = 1 if X ≠ 0
(A2)	If X = 0, then $\overline{X}$ = 1	(A2')	If X = 1, then $\overline{X}$ = 0
(A3)	0 • 0 = 0	(A3')	1 + 1 = 1
(A4)	1 • 1 = 1	(A4')	0 + 0 = 0
(A5)	1 • 0 = 0 • 1 = 0	(A5')	0 + 1 = 1 + 0 = 1

Duality

- Every truth statement can yields another truth statement
  - I *exercise* if I have *time and energy* *(original statement)*
  - I *don't exercise* if I *don't have time or don't have energy* *(dual statement)*
- To express the dual, swap…

__ ⟺ __

__ ⟺ __

Duality

- The "dual" of an expression is not equal to the original

$$1 + 0 \neq 0 \bullet 1$$

Original expression **Dual**

- Taking the "dual" of both sides of an equation yields a new equation

$$X + 1 = 1 \Rightarrow X \bullet 0 = 0$$

Original equation **Dual**

Single Variable Theorems

- Provide some simplifications for expressions containing:
  - a single variable
  - a single variable and a constant bit
- Each theorem has a dual (another true statement)
- Each theorem can be proved by writing a truth table for both sides (i.e. proving the theorem holds for all possible values of X)

T1	**X + 0 = X**	**T1'**	**X • 1 = X**
T2	**X + 1 = 1**	**T2'**	**X • 0 = 0**
T3	**X + X = X**	**T3'**	**X • X = X**
T4	**(X')' = X**		
T5	**X + X' = 1**	**T5'**	**X • X' = 0**

Single Variable Theorem (T1)

X+0 = X (T1)

X•1 = X (T1')

Hold Y constant

OR

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

AND

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

Whenever a variable is OR'ed with 0, the output will be the same as the variable…

"0 OR Anything equals that ____________"

Whenever a variable is AND'ed with 1, the output will be the same as the variable…

"1 AND Anything equals that ____________"

Single Variable Theorem (T2)

X+1 = 1 (T2)

X•0 = 0 (T2')

Hold Y constant

OR

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

AND

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

Whenever a variable is OR'ed with 1, the output will be 1…

"1 OR anything equals __________"

Whenever a variable is AND'ed with 0, the output will be 0…

"0 AND anything equals __________"

Single Variable Theorem (T3)

X+X = X (T3)

X•X = X (T3')

OR

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

AND

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

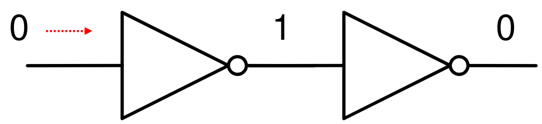
Whenever a variable is OR'ed with itself, the result is just the value of the variable

Whenever a variable is AND'ed with itself, the result is just the value of the variable

This theorem can be used to reduce two identical terms into one *OR* to replicate one term into two.

Single Variable Theorem (T4)

$(X')' = X$ (T4) $(\overline{\overline{X}}) = X$ (T4)



Anything inverted twice yields its original value

Single Variable Theorem (T5)

$X+\overline{X} = 1$ (T5) $X \bullet \overline{X} = 0$ (T5')

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

OR

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

AND

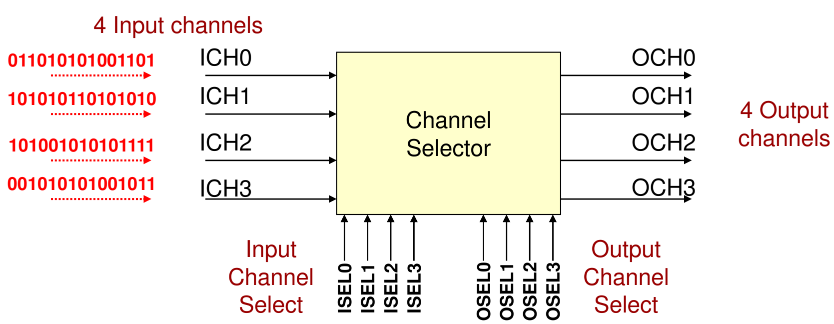
Whenever a variable is OR'ed with its complement, one value has to be 1 and thus the result is 1

Whenever a variable is AND'ed with its complement, one value has to be 0 and thus the result is 0

This theorem can be used to simplify variables into a constant or to expand a constant into a variable.

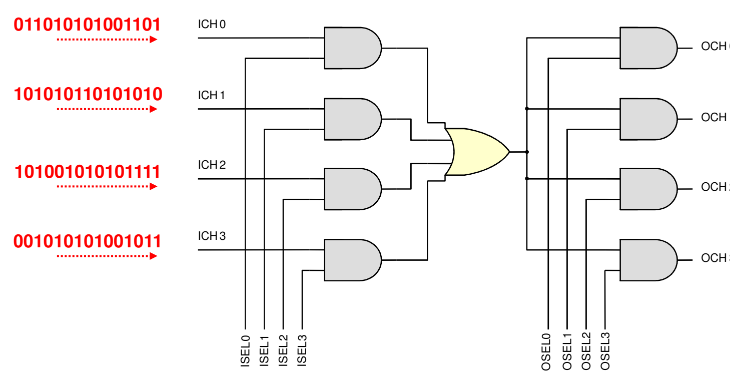
Application: Channel Selector

- Given 4 input, digital music/sound channels and 4 output channels
- Given individual "select" inputs that select 1 input channel to be routed to 1 output channel



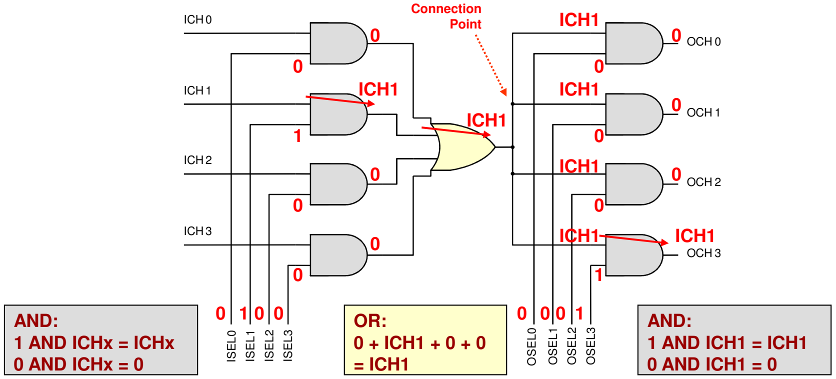
Application: Steering Logic

- 4-input music channels (ICHx)
  - Select one input channel (use ISELx inputs)
  - Route to one output channel (use OSELx inputs)



Application: Steering Logic

- 1st Level of AND gates act as barriers only passing 1 channel
- OR gates combines 3 streams of 0's with the 1 channel that got passed (i.e. ICH1)
- 2nd Level of AND gates passes the channel to only the selected output



Your Turn

- Build a circuit that takes 3 inputs: S, IN0, IN1 and outputs a single bit Y.
- It's functions should be:
  - If S = 0, Y = IN0 (IN0 passes to Y)
  - If S = 1, Y = IN1 (IN1 passes to Y)