# Unit 10

Signed Representation Systems

Binary Arithmetic



# BINARY REPRESENTATION SYSTEMS REVIEW



# Interpreting Binary Strings

- Given a string of 1's and 0's, you need to know the *representation system* being used, before you can understand the value of those 1's and 0's.
- Information (value) = Bits + Context (System)

01000001 = ?




# Binary Representation Systems

- Integer Systems
  - Unsigned
    - Unsigned (Normal) binary
  - Signed
    - Signed Magnitude
    - 2's complement
    - *Excess-N**
    - *1's complement**
- Floating Point
  - For very large and small (fractional) numbers
- Codes
  - Text
    - ASCII / Unicode
  - Decimal Codes
    - BCD (Binary Coded Decimal) / (8421 Code)

* = Not fully covered in this class

# Review of Number Systems

- Number systems consist of
  1. A base (radix) r
  2. r coefficients [0 to r-1]
- Human System: Decimal (Base 10): 0,1,2,3,4,5,6,7,8,9
- Computer System: Binary (Base 2): 0,1
- Human systems for working with computer systems (shorthand for human to read/write binary)
  - Octal (Base 8): 0,1,2,3,4,5,6,7
  - Hexadecimal (Base 16): 0-9,A,B,C,D,E,F (A thru F = 10 thru 15)



# Binary Examples

$$(\underset{8}{\underline{1}}\underset{4}{\underline{0}}\underset{2}{\underline{0}}\underset{1}{\underline{1}}.\underset{.5}{\underline{1}})_2 = 8 + 1 + 0.5 = 9.5_{10}$$

$$(\underset{128}{\underline{1}}0\underset{32}{\underline{1}}\underset{16}{\underline{1}}000\underset{1}{\underline{1}})_2 = 128 + 32 + 16 + 1 = 177_{10}$$



# Unique Combinations

- Given *n* digits of base *r*, how many unique numbers can be formed? $r^n$
  - What is the range? [0 to $r^n$-1]

2-digit, decimal numbers (r=10, n=2) ___ ___ (0-9 0-9) — **100 combinations: 00-99**

3-digit, decimal numbers (r=10, n=3) ___ ___ ___ — **1000 combinations: 000-999**

4-bit, binary numbers (r=2, n=4) ___ ___ ___ ___ (0-1 0-1 0-1 0-1) — **16 combinations: 0000-1111**

6-bit, binary numbers (r=2, n=6) ___ ___ ___ ___ ___ ___ — **64 combinations: 000000-111111**

Main Point: Given n digits of base r, $r^n$ unique numbers can be made with the range [0 - ($r^n$-1)]



# Approximating Large Powers of 2

- Often need to find decimal approximation of a large powers of 2 like $2^{16}$, $2^{32}$, etc.
- Use following approximations:
  - $2^{10} \approx 10^3$ (1 thousand) = 1 Kilo-
  - $2^{20} \approx 10^6$ (1 million) = 1 Mega-
  - $2^{30} \approx 10^9$ (1 billion) = 1 Giga-
  - $2^{40} \approx 10^{12}$ (1 trillion) = 1 Tera-
- For other powers of 2, decompose into product of $2^{10}$ or $2^{20}$ or $2^{30}$ and a power of 2 that is less than $2^{10}$
  - 16-bit half word: 64K numbers
  - 32-bit word: 4G numbers
  - 64-bit dword: 16 million trillion numbers

**$2^{16} = 2^6 * 2^{10} \approx 64 * 10^3 = 64{,}000$**

**$2^{24} = 2^4 * 2^{20} \approx 16 * 10^6 = 16{,}000{,}000$**

**$2^{28} = 2^8 * 2^{20} \approx 256 * 10^6 = 256{,}000{,}000$**

**$2^{32} = 2^2 * 2^{30} \approx 4 * 10^9 = 4{,}000{,}000{,}000$**

# Decimal to Unsigned Binary

- To convert a decimal number, *x,* to binary:
  - Only coefficients of 1 or 0. So simply find place values that add up to the desired values, starting with larger place values and proceeding to smaller values and place a 1 in those place values and 0 in all others

| | | | | | | |
|---|---|---|---|---|---|---|
| $25_{10}$ = | 0 | 1 | 1 | 0 | 0 | 1 |
| | 32 | 16 | 8 | 4 | 2 | 1 |

**For $25_{10}$ the place value 32 is too large to include so we include 16. Including 16 means we have to make 9 left over. Include 8 and 1.**



# Decimal to Another Base

- To convert a decimal number, *x,* to base r:
  - Use the place values of base r (powers of r). Starting with largest place values, fill in coefficients that sum up to desired decimal value without going over.

| | | | | |
|---|---|---|---|---|
| $75_{10}$ = | 0 | 4 | B | hex |
| | 256 | 16 | 1 | |



Signed Magnitude

2's Complement System

# SIGNED SYSTEMS



# Binary Representation Systems

- Integer Systems
  - Unsigned
    - Unsigned (Normal) binary
  - Signed
    - Signed Magnitude
    - 2's complement
    - *1's complement**
    - *Excess-N**
- Floating Point
  - For very large and small (fractional) numbers
- Codes
  - Text
    - ASCII / Unicode
  - Decimal Codes
    - BCD (Binary Coded Decimal) / (8421 Code)

* = Not covered in this class

# Unsigned and Signed

- Normal (unsigned) binary can only represent positive numbers
  - All place values are positive
- To represent negative numbers we must use a modified binary representation that takes into account sign (pos. or neg.)
  - We call these *signed* representations



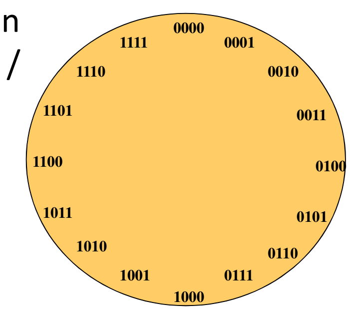
# Signed Number Representation

- 2 Primary Systems
  - ________________________
  - ________________________ *(most widely used for integer representation)*



# Signed numbers

- All systems used to represent negative numbers split the possible binary combinations in half (half for positive numbers / half for negative numbers)
- In both signed magnitude and 2's complement, positive and negative numbers are separated using the MSB
  - __________ means negative
  - __________ means positive






# Signed Magnitude System

- Use binary place values but now MSB represents the sign (1 if negative, 0 if positive)

**4-bit Unsigned**

| Bit 3 | Bit 2 | Bit 1 | Bit 0 |
|---|---|---|---|
| ___ | ___ | ___ | ___ |
| 8 | 4 | 2 | 1 |

⇒ **0 to 15**

**4-bit Signed Magnitude**

| Bit 3 | Bit 2 | Bit 1 | Bit 0 |
|---|---|---|---|
| ___ | ___ | ___ | ___ |
| +/- | 4 | 2 | 1 |

⇒ **-7 to +7**

**8-bit Signed Magnitude**

| Bit 7 | Bit 6 | Bit 5 | Bit 4 | Bit 3 | Bit 2 | Bit 1 | Bit 0 |
|---|---|---|---|---|---|---|---|
| ___ | ___ | ___ | ___ | ___ | ___ | ___ | ___ |
| +/- | 64 | 32 | 16 | 8 | 4 | 2 | 1 |

⇒ **-127 to +127**

# Signed Magnitude Examples

**4-bit Signed Magnitude**

| | | | | |
|---|---|---|---|---|
| **1** | **1** | **0** | **1** | **= -5** |
| +/- | 4 | 2 | 1 | |
| **0** | **0** | **1** | **1** | **= +3** |
| +/- | 4 | 2 | 1 | |
| **1** | **1** | **1** | **1** | **= -7** |
| +/- | 4 | 2 | 1 | |

Notice that +3 in signed magnitude is the same as in the unsigned system

**8-bit Signed Magnitude**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **1** | **0** | **0** | **1** | **0** | **0** | **1** | **1** | **= -19** |
| +/- | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |
| **0** | **0** | **0** | **1** | **1** | **0** | **0** | **1** | **= +25** |
| +/- | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

Important: Positive numbers have the same representation in signed magnitude as in normal unsigned binary



# Signed Magnitude Range

- Given n bits…
  - MSB is sign
  - Other n-1 bits = normal unsigned place values
    - Range with n-1 unsigned bits = [0 to $2^{n-1}$-1]

Range with n-bits of Signed Magnitude

[ $-2^{n-1}-1$ to $+2^{n-1}-1$]



# Disadvantages of Signed Magnitude

1. Wastes a combination to represent _____

   0000 = 1000 = ____

2. Addition and subtraction algorithms for signed magnitude are different than unsigned binary (we'd like them to be the same to use same HW)

```
   4          6
 – 6        – 4
 ___        ___
Swap &       –
make res.
negative
```



# 2's Complement System

- Normal binary place values except MSB has ____________________
  - MSB of 1 = ______

**4-bit Unsigned**

| Bit 3 | Bit 2 | Bit 1 | Bit 0 | |
|---|---|---|---|---|
| ___ | ___ | ___ | ___ | ⇨ **0 to 15** |
| 8 | 4 | 2 | 1 | |

**4-bit 2's complement**

| Bit 3 | Bit 2 | Bit 1 | Bit 0 | |
|---|---|---|---|---|
| ___ | ___ | ___ | ___ | ⇨ **-8 to +7** |
| __ | 4 | 2 | 1 | |

**8-bit 2's complement**

| Bit 7 | Bit 6 | Bit 5 | Bit 4 | Bit 3 | Bit 2 | Bit 1 | Bit 0 | |
|---|---|---|---|---|---|---|---|---|
| ___ | ___ | ___ | ___ | ___ | ___ | ___ | ___ | ⇨ **-128 to +127** |
| ___ | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

# 2's Complement Examples

**4-bit 2's complement**

| 1 | 0 | 1 | 1 | = -5 |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | |

| 0 | 0 | 1 | 1 | = +3 |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | |

Notice that +3 in 2's comp. is the same as in the unsigned system

| 1 | 1 | 1 | 1 | = -1 |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | |

**8-bit 2's complement**

| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | = -127 |
|---|---|---|---|---|---|---|---|---|
| -128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | = +25 |
|---|---|---|---|---|---|---|---|---|
| -128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

Important: Positive numbers have the _______ representation in 2's complement as in normal unsigned binary



# 2's Complement Range

- Given n bits…
  - Max positive value = ______________
    - Includes all n-1 positive place values
  - Max negative value = __________________
    - Includes only the negative MSB place value

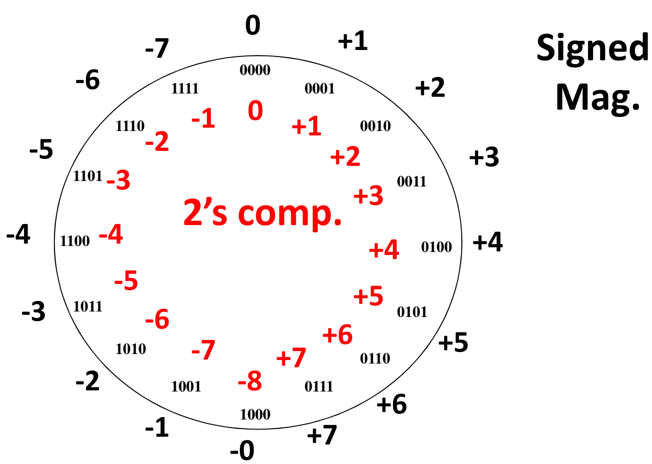
Range with n-bits of 2's complement

$$[\ -2^{n-1} \text{ to } +2^{n-1}-1]$$

  - Side note – What decimal value is 111…11?



# Comparison of Systems






# Unsigned and Signed Variables

- In C, _______ variables use unsigned binary (normal power-of-2 place values) to represent numbers

| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | = +147 |
|---|---|---|---|---|---|---|---|---|
| 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

- In C, signed variables use the _________________ (Neg. MSB weight) to represent numbers

| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | = -109 |
|---|---|---|---|---|---|---|---|---|
| -128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 | |

# IMPORTANT NOTE

- All computer systems use the **2's complement system** to represent **signed integers**!
- So from now on, if we say an integer is **signed**, we are actually saying it uses the 2's complement system unless otherwise specified
  - We will not use "signed magnitude" unless explicitly indicated



# Zero and Sign Extension

- Extension is the process of increasing the number of bits used to represent a number without changing its value

Unsigned = Zero Extension (Always add leading ___'s):

111011 = ___111011

Increase a 6-bit number to 8-bit number by _____ extending

2's complement = Sign Extension (Replicate _____ bit):

pos. 011010 = ___011010

neg. 110011 = ___110011

_____ bit is just repeated as many times as necessary



# Zero and Sign Truncation

- Truncation is the process of decreasing the number of bits used to represent a number without changing its value

Unsigned = Zero Truncation (Remove leading 0's):

~~00~~111011 = 111011

Decrease an 8-bit number to 6-bit number by truncating 0's. Can't remove a '1' because value is changed

2's complement = Sign Truncation (Remove _______ of sign bit):

pos. 00011010 = _________

neg. 11110011 = _________

Any copies of the MSB can be removed without changing the numbers value. Be careful not to change the sign by cutting off ALL the sign bits.



# Data Representation

- In C/C++ variables can be of different types and sizes
  - Integer Types (signed and unsigned)

| C Type | Bytes | Bits | ATmega328 |
|---|---|---|---|
| [unsigned] char | 1 | 8 | byte |
| [unsigned] short [int] | 2 | 16 | word |
| [unsigned] long [int] | 4 | 32 | -[1] |
| [unsigned] long long [int] | 8 | 64 | -[1] |

[1]Can emulate but has no single-instruction support

  - Floating Point Types

| C Type | Bytes | Bits | ATmega328 |
|---|---|---|---|
| float | 4 | 32 | N/A[1] |
| double | 8 | 64 | N/A[1] |

# ARITHMETIC



# Binary Arithmetic

- Can perform all arithmetic operations (+,-,*,÷) on binary numbers
- Can use same methods as in decimal
  - Still use carries and borrows, etc.
  - Only now we carry when sum is 2 or more rather than 10 or more (decimal)
  - We borrow 2's not 10's from other columns
- Easiest method is to add bits in your head in decimal (1+1 = 2) then convert the answer to binary ($2_{10} = 10_2$)



# Binary Addition

- In decimal addition we ______ when the sum is 10 or more
- In binary addition we carry when the sum is __ or more
- Add bits in binary to produce a sum bit and a carry bit

```
   0         0         1        1
                                1
 + 0       + 1       + 0      + 1
 ---       ---       ---      ---
  00        01        01       10
```

no need to carry, sum bit

no need to carry, sum bit

no need to carry, sum bit

carry 1 into next column of bits, sum bit



# Binary Addition & Subtraction

```
    0 1 1 1          1 0 1 0
  + 0 0 1 1        - 0 1 0 1
  ---------        ---------
```

# Binary Addition

```
  0110 (6)
  8 4 2 1
+ 0111 (7)
------
       (13)
```



# Binary Addition

1

```
     0
  0110 (6)
+ 0111 (7)
------
  1101 (13)
```

```
  0
+ 1
---
 01
```

carry bit
sum bit

2

```
    10
  0110 (6)
+ 0111 (7)
------
  1101 (13)
```

```
  0
  1
+ 1
---
 10
```

carry bit
sum bit

3

```
   110
  0110 (6)
+ 0111 (7)
------
  1101 (13)
```

```
  1
  1
+ 1
---
 11
```

carry bit
sum bit

4

```
   110
  0110 (6)
+ 0111 (7)
------
  1101 (13)
```

```
  1
  0
+ 0
---
 01
```

carry bit
sum bit



# Hexadecimal Arithmetic

- Same style of operations
  - Carry when sum is 16 or more, etc.

$$\begin{array}{r} 4\ D_{16} \\ +\ B\ 5_{16} \\ \hline \end{array}$$

__ __
16 1

__ __
16 1



"Taking the 2's complement"

## SUBTRACTION THE EASY WAY

# Taking the Negative

- Given a number in signed magnitude or 2's complement how do we find its negative (i.e. -1 * X)
  - Signed Magnitude: _______________
    - 0110 = +6 => _____________
  - 2's complement: "___________________________"
    - 0110 = +6 => _________
    - Operation defined as:
      1. Flip/invert/not _________ (1's complement)
      2. Add ___ and drop any _______ (i.e. finish with the same # of bits as we start with)



# Taking the 2's Complement

- Invert (flip) each bit (take the 1's complement)
  - 1's become 0's
  - 0's become 1's
- Add 1 (drop final carry-out, if any)

-32 16 8 4 2 1

010011

Original number = +19

Bit flip is called the 1's complement of a number

Resulting number = -19

Important: Taking the 2's complement is equivalent to taking the negative (negating)



# Taking the 2's Complement

1

-32 16 8 4 2 1

101010

Original number = ____

Take the 2's complement yields the negative of a number

Resulting number = ____

Back to original = ____

2

0000

Original # = 0

Take the 2's complement

2's comp. of 0 is __

3

1000

Original # = -8

Take the 2's complement

Negative of -8 is __

(i.e. no positive equivalent, but this is not a huge problem)



# 2's Complement System Facts

- Normal binary place values but MSB has negative weight
- MSB determines sign of the number
  - 0 = positive / 1 = negative
- Special Numbers
  - 0 = All 0's (00…00)
  - -1 = All 1's (11…11)
  - Max Positive = 0 followed by all 1's (011..11)
  - Max Negative = 1 followed by all 0's (100…00)
- To take the negative of a number (e.g. -7 => +7 or +2 => -2), requires taking the complement
  - 2's complement of a # is found by flipping bits and adding 1

| | |
|---|---|
| 1001 | x = -7 |
| 0110 | Bit flip (1's comp.) |
| + 1 | Add 1 |
| 0111 | -x = -(-7) = +7 |

## ADDITION AND SUBTRACTION



# 2's Complement Addition/Subtraction

- Addition
  - Sign of the numbers do not matter
  - ____________________
  - ____________________
- Subtraction
  - Any subtraction (A-B) can be converted to addition (__________) by taking the ___________________ of B
  - (A-B) becomes (______________________)
  - Drop any carry-out



# 2's Complement Addition

- No matter the sign of the operands just add as normal
- Drop any extra carry out

```
  0011        1101
+ 0010      + 0010
------      ------



  0011        1101
+ 1110      + 1110
------      ------
```



# Unsigned and Signed Addition

- Addition process is the same for both unsigned and signed numbers
  - Add columns right to left
- Examples:

```
            If unsigned  If signed
  1001
+ 0011
------
```

# 2's Complement Subtraction

- Take the 2's complement of the subtrahend and add to the original minuend
- Drop any extra carry out

```
   0011 (+3)
 - 0010 (+2)
 ------
```

```
   1101 (-3)
 - 1110 (-2)
 ------
```



# Unsigned and Signed Subtraction

- Subtraction process is the same for both unsigned and signed numbers
  - Convert A – B to A + Comp. of B
  - Drop any final carry out
- Examples:

|  | If unsigned | If signed |
|---|---|---|
| 1100 | (12) | (-4) |
| - 0010 | (2) | (2) |

If unsigned If signed



# Important Note

- Almost all computers use 2's complement because…
- The same addition and subtraction ____________ can be used on unsigned and 2's complement (signed) numbers
- Thus we only need one _________________________ to perform operations on both unsigned and signed numbers



# OVERFLOW

# Overflow

- Overflow occurs when the result of an arithmetic operation is ______________ ___________________________________
- Conditions and tests to determine overflow depend on __________

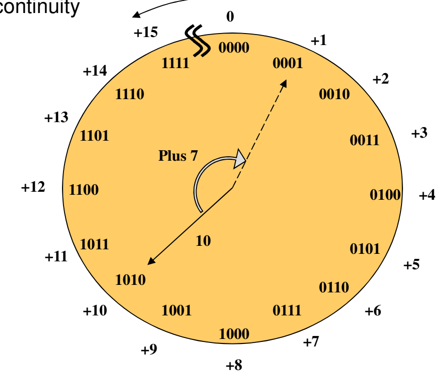


# Unsigned Overflow

Overflow occurs when you cross this discontinuity

**10 + 7 = 17**

With 4-bit *unsigned* numbers we can only represent 0 – 15. Thus, we say overflow has occurred.



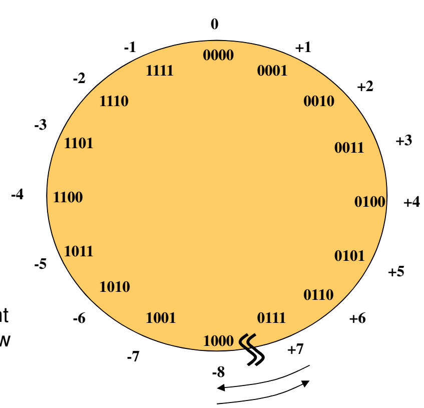



# 2's Complement Overflow

5 + 7 = +12

-6 + -4 = -10

With 4-bit *2's complement* numbers we can only represent -8 to +7. Thus, we say overflow has occurred.




Overflow occurs when you cross this discontinuity



# Overflow in Addition

- Overflow occurs when the result of the addition cannot be represented with the given number of bits.
- Tests for overflow:
  - Unsigned: ______________
  - Signed: ____________________

| | If unsigned | If signed |
|---|---|---|
| 1101 | | |
| + 0100 | | |
| 0001 | | |

| | If unsigned | If signed |
|---|---|---|
| 0110 | | |
| + 0101 | | |
| 1011 | | |

# Overflow in Subtraction

- Overflow occurs when the result of the subtraction cannot be represented with the given number of bits.
- Tests for overflow:
  - Unsigned: if Cout = 0
  - Signed: if addition is p + p = n or n + n = p

**If unsigned** **If signed**

```
  0111
- 1000
```



# FLOATING POINT



# Floating Point

- Used to represent very small numbers (fractions) and very large numbers
  - Avogadro's Number: $+6.0247 * 10^{23}$
  - Planck's Constant: $+6.6254 * 10^{-27}$
  - Note: 32 or 64-bit integers can't represent this range
- Floating Point representation is used in HLL's like C by declaring variables as `float` or `double`



# Fixed Point

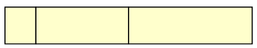
- Unsigned and 2's complement fall under a category of representations called "Fixed Point"
- The radix point is assumed to be in a fixed location for all numbers [Note: we could represent fractions by implicitly assuming the binary point is at the _________. Variables just store bits…you can assume the binary point is anywhere you like]
  - Integers: 10011101. (binary point to right of LSB)
    - For 32-bits, unsigned range is 0 to ~4 billion
  - Fractions: .10011101 (binary point to left of MSB)
    - Range [0 to 1)
- **Main point**: By ___________ the radix point, we limit the range of numbers that can be represented
  - Floating point allows the radix point to be in a different location for each value

**Bit storage**

**Fixed point Rep.**

# Floating Point Representation

- Similar to scientific notation used with decimal numbers
  - ±D.DDD * 10 $^{\pm exp}$
- Floating Point representation uses the following form
  - ___________________
  - 3 Fields: _____, __________, ______________ (also called _____________________)





# Normalized FP Numbers

- Decimal Example
  - +0.754*$10^{15}$ is not correct scientific notation
  - Must have exactly _____ significant digit before decimal point: _______________
- In binary the only _______________is '1'
- Thus normalized FP format is:

  _______________
- FP numbers will **always be normalized** before being __________ in memory or a reg.
  - The *1.* is actually ___________ but assumed since we always will store normalized numbers
  - If HW calculates a result of 0.001101*$2^5$ it must normalize to 1.101000*$2^2$ before storing



# IEEE Floating Point Formats

- Single Precision (32-bit format)
  - 1 Sign bit (0=pos/1=neg)
  - __ Exponent bits
    - _________ representation
    - More on next slides
  - ___ fraction (significand or mantissa) bits
  - Equiv. Decimal Range:
    - 7 digits x $10^{\pm 38}$

| 1 | 8 | 23 |
|---|---|---|
| S | Exp. | Fraction |

- Double Precision (64-bit format)
  - 1 Sign bit (0=pos/1=neg)
  - ___ Exponent bits
    - _________ representation
    - More on next slides
  - ___ fraction (significand or mantissa) bits
  - Equiv. Decimal Range:
    - 16 digits x $10^{\pm 308}$

| 1 | 11 | 52 |
|---|---|---|
| S | Exp. | Fraction |



# Floating Point vs. Fixed Point

- Single Precision (32-bits) Equivalent Decimal Range:
  - 7 significant decimal digits * $10^{\pm 38}$
  - Compare that to 32-bit signed integer where we can represent ±2 billion. How does a 32-bit float allow us to represent such a greater range?
  - FP allows for range but sacrifices precision (can't represent all numbers in its range)
- Double Precision (64-bits) Equivalent Decimal Range:
  - 16 significant decimal digits * $10^{\pm 308}$

# Exponent Representation

- Exponent needs its own sign (+/-)
- Rather than using 2's comp. system we use Excess-N representation
  - Single-Precision uses Excess-127
  - Double-Precision uses Excess-1023
  - This representation allows FP numbers to be easily compared
- Let E' = stored exponent code and E = true exponent value
- For single-precision: E' = E + 127
  - $2^1$ => E = 1, E' = $128_{10}$ = $10000000_2$
- For double-precision: E' = E + 1023
  - $2^{-2}$ => E = -2, E' = $1021_{10}$ = $01111111101_2$

| 2's comp. | E' (stored Exp.) | Excess-127 |
|---|---|---|
| | 1111 1111 | |
| | 1111 1110 | |
| | | |
| | 1000 0000 | |
| | 0111 1111 | |
| | 0111 1110 | |
| | | |
| | 0000 0001 | |
| | 0000 0000 | |

**Comparison of 2's comp. & Excess-N**

**Q: Why don't we use Excess-N more to represent negative #'s**



# Single-Precision Examples

1

$2^7$=128 $2^1$=2

| 1 | 1000 0010 | 110 0110 0000 0000 0000 0000 |
|---|---|---|

2

+0.6875 = +0.1011